LECTURE 30

Series Solutions to Differential Equations

Sometimes, if we can't find a nice solution to a differential equations, we can still find a series representation of the solution. This is still very useful: if we know that the series converges, we can approximate the solution as closely as we want. This is similar (but not quite the same) as finding the Taylor series of a function.

1. Power Series Review

We should first review some facts about power series. Recall that a power series has the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for some x_0 and some coefficients a_n. We can strip out as many terms as we like; for example, the following are all equation to $f(x)$.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &= a_0 + \sum_{n=1}^{\infty} a_n (x - x_0)^n \\ &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \sum_{n=3}^{\infty} a_n (x - x_0)^n \end{aligned}$$

Further, a power series can be reindexed to start at any initial index value by shifting the index in the sum. Suppose we start with the power series

$$\sum_{n=2}^{\infty} (n-1) a_n x^{n+3},$$

but we want it to start at $n = 0$ instead of $n = 2$. To do this, we start by defining $m = n - 2$. Then $n = m + 2$, and we can rewrite the series in terms of m:

$$\begin{aligned} \sum_{n=2}^{\infty} (n-1) a_n x^{n+2} &= \sum_{m=0}^{\infty} (m + 2 - 1) a_{m+2} x^{m+2+3} \\ &= \sum_{m=0}^{\infty} (m+1) a_{m+2} x^{m+5}. \end{aligned}$$

Then, since the index is really nothing more than a dummy variable, we can just replace m with n to get that the original sum is

$$\sum_{n=0}^{\infty} (n+1) a_{n+2} x^{n+5}.$$

Once you get comfortable doing this, you can see that all that we need was replace all instances of n by $n+2$, which allowed us to start n at 0 instead of 2, and you don't need to do the intermediate work.

Index shifts are important for adding or subtracting two power series together. If we have two power series, adding them is quite simple, conceptually, since a power series is just a special type of infinite sum. However, for the formulas to be consolidated into a single sum, we would need the two series to start at the same index value, and for each index value to correspond to the same power of $(x - x_0)$.

Example 30.1. *Write the following as a single power series.*

$$\sum_{n=0}^{\infty} n a_n (x-2)^{n+1} + \sum_{n=2}^{\infty} n^2 a_n (x-2)^n.$$

There are two things that need to be done here: first, we need the exponents of $x-2$ to match up, and second, the two series need to start at the same value of n. This will require us to shift the index of the first sum down by 1, so that the exponent is n, rather than $n+1$. This will make the sum start at $n = 1$, rather than $n = 0$. Doing so, we get

$$\sum_{n=1}^{\infty} (n-1) a_{n-1} (x-2)^n + \sum_{n=2}^{\infty} n^2 a_n (x-2)^n.$$

Now, we need the two series to start at the same value of n. We can't do any index shifts, but we can strip out the first term (corresponding to $n = 1$) in the first sum, so that the two big sums will start at $n = 2$. In this case, that first term is actually 0, since the coefficient is $1 - 1$, but in general (as we will see when working with these), this may leave us with terms out in front of the sum. Again, though, in this case that term is 0, so we just have that our sum is

$$\sum_{n=2}^{\infty} \left[(n-1) a_{n-1} + n^2 a_n\right] (x-2)^n.$$

□

Differentiating a power series is also easy; it can be done term by term. That is, if

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-x_0)^n \\ &= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots, \end{aligned}$$

we have

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots \\ &= \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}. \end{aligned}$$

Notice as well that we could've started this last sum at $n = 0$ without having to shift indices, since the n out front would make that term 0 anyway.

1.1. Convergence. Whenever we deal with infinite series, we have to be worried about whether the infinite sum actually adds up to anything or if it grows without bound, and in the case of a power series, we also have to be worried about where the sum will converge or diverse. A power series *converges* at $x = c$ if the limit of partial sums

$$\lim_{N\to\infty}\sum_{n=0}^{N} a_n(c - x_0)^n$$

exists. Any power series will always converge at $x = x_0$, so any given power series will always converge at at least one value of x. How many other points the series converges at is captured by the *radius of convergence,* which is the (possibly infinite) value $0 \le \rho \le \infty$ such that the series will converge for any x such that $|x - x_0| < \rho$ and the series will converge for any x such that $|x - x_0| > \rho$. Whether the series converges or not for a given x precisely ρ away from x_0 has to be determined on a case by case basis.

The most common way to determine the radius of convergence for a series is to use the *ratio test.* For this, we compute

$$L = |x - x_0| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

If $L < 1$, the series will converge, if $L > 1$, it will diverse, and if $L = 1$, the test fails and we need another method.

If you need more details on power series at this point, please look at a calculus reference.

2. Series Solutions

Now suppose we have the homogeneous linear second order differential equation is

$$p(x)y'' + q(x)y' + r(x) = 0. \tag{30.1}$$

Definition 30.2. We say that a point $x = x_0$ is an *ordinary point* if the functions $\frac{q(x)}{p(x)}$ and $\frac{r(x)}{p(x)}$ are *analytic at* x_0, that is, they have Taylor series expansions at x_0 with a positive radius of convergence and which converge appropriately to the original functions.

A point which is not an ordinary point is called a *singular point.*

It is generally the case that singular points occur where $p(x) = 0$, but as the next example will show, this is not always true.

Example 30.3. The point $x = 0$ is an ordinary point of the equation

$$xy'' + \sin(x)y' + x^2y = 0,$$

since $\frac{\sin(x)}{x}$ and x are analytic at $x = 0$. On the other hand, $x = 0$ is a singular point for

$$xy'' + y' + xy = 0.$$

□

It will turn out that we can only find series solutions to our differential equations near ordinary points, but in practice, this is not as big of an obstacle as it may seem, since for any equation we'll be looking at, there aren't very many singular points (we won't quite get into what this means, technically). What we do is to suppose that we have a series representation,

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0), \tag{30.2}$$

and then we can figure out what coefficients a_n need to be by differentiating (30.2) and plugging the derivatives into (30.1). Once we have the appropriate coefficients, we say that (30.2) is the *series solution to* (30.1) *near* $x = x_0$. It's good within the radius of convergence.

Let's demonstrate the method by using a couple of basic examples with constant coefficients.

Example 30.4. *Determine a series solution to*

$$y'' - y = 0.$$

First, we observe that $p(x) = 1$, and so the function we need to be analytic near $x = 0$ is $q(x) = 1$, which definitely is. We suppose that the series solution has the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating this, we have

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Plugging these into the differential equation, we have

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

We want to write the left hand side as a single series; this will first involve reindexing the first sum so that x has the same exponent in each sum.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

In general, to combine these sums, we would need to pull out any extra terms so that the sums started at the same value of n. In this case, that's unnecessary, so we can proceed.

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n\right] x^n = 0.$$

Now, recall that if a power series is always equal to 0, each coefficient must be as well. Thus we have

$$(n+2)(n+1) a_{n+2} - a_n = 0. \tag{30.3}$$

This is an example of a *recurrence relation,* and we would like to use it to figure out what each a_n is. We can't do that directly, but we can use (30.3) to solve for a_{n+2} in terms of a_n. This will allow us to write down each a_n in terms of a_0 and a_1, recursively. We get

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)}. \tag{30.4}$$

To use this, we will need to figure out what this tells us about specific values of a_n. The most convenient way to do that is to start plugging in values of n and see what patterns emerge.

$$
\begin{aligned}
n &= 0 & a_2 &= \frac{a_0}{(2)(1)} & n &= 1 & a_3 &= \frac{a_1}{(3)(2)} \\
n &= 2 & a_4 &= \frac{a_2}{(4)(3)} & n &= 3 & a_5 &= \frac{a_3}{(5)(4)} \\
& & &= \frac{a_0}{(4)(3)(2)(1)} & & & &= \frac{a_1}{(5)(4)(3)(2)} \\
n &= 4 & a_6 &= \frac{a_4}{((6)(5)} & n &= 5 & a_7 &= \frac{a_5}{(7)(6)} \\
& & &= \frac{a_0}{((6)(5)(4)(3)(2)(1)} & & & &= \frac{a_1}{(7)(6)(5)(4)(3)(2)(1)} \\
& & &\vdots & & & &\vdots \\
& & a_{2k} &= \frac{a_0}{(2k)!} & & & a_{2k+1} &= \frac{a_1}{(2k+1)!}
\end{aligned}
$$

Now, we can plug these into the series and collect the a_0 terms and the a_1 terms:

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_{2k} x^{2k} + a_{2k+1} x^{2k+1} + \ldots \\
&= a_0 + a_1 x + \frac{a_0}{2!} x^2 + \frac{a_1}{3!} x^3 + \ldots + \frac{a_0}{(2k)!} x^{2k} + \frac{a_1}{(2k+1)!} x^{2k+1} + \ldots \\
&= a_0 \left(1 + \frac{x^2}{2!} + \ldots + \frac{x^{2k}}{(2k)!} + \ldots\right) + a_1 \left(x + \frac{x^3}{3!} + \ldots + \frac{x^{2k+1}}{(2k+1)!} + \ldots\right) \\
&= a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!},
\end{aligned}
$$

□

There is one important remark to make about Example 30.4. We already know that the general solution to the differential equation $y'' - y = 0$ is

$$y(x) = c_1 e^x + c_2 e^{-x}.$$

The series solution that we obtained should correspond to this. If we Taylor expand this general solution, we get:

$$
\begin{aligned}
y(x) &= c_1 \sum_{n=0}^{\infty} \frac{x^n}{n!} + c_2 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \\
&= c_1 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\right) + c_2 \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right) \\
&= (c_1 + c_2)1 + (c_1 - c_2)x + \ldots + (c_1 + c_2)\frac{x^{2k}}{(2k)!} + (c_1 - c_2)\frac{x^{2k+1}}{(2k+1)!} + \ldots,
\end{aligned}
$$

and setting $a_0 = c_1 + c_2$ and $a_1 = c_1 - c_2$ and simplifying, we get

$$= a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

Thus we see that these two solutions are actually the same, as they should be, and that a choice of initial data would let us find a_0 and a_1.

Example 30.5. *Compute a series solution around $x = 0$ for the differential equation*

$$y'' - xy = 0.$$

Notice that, once again, every point is an ordinary point, so $x = 0$ in particular is no problem. We have:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Plugging in gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0.$$

We need to move the coefficient in front of the second term into the series so we can consolidate them; this is easy in this case, since we can just distribute it through and it combines nicely with the x^n term.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

The first step to combining these sums is to get the same exponent for x, which we do by shifting the first series down by 2 and the second up by 1.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

We can't combine these yet, since they don't start at the same value of n; the only way to do that is to explicitly write the term in the first series corresponding to $n = 0$, so that the sum starts at $n = 1$.

$$(2)(1) a_2 x^0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$
$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1}\right] x^n = 0.$$

Now, we know that each term in this sum has to be equal to 0, remembering that the first term out front is the $n = 0$ term. So we have

$$\begin{aligned} n = 0: &\quad 2a_2 = 0 &\Rightarrow\quad & a_2 = 0 \\ n > 0: &\quad (n+2)(n+1)a_{n+2} - a_{n-1} = 0 &\Rightarrow a_{n+2} \quad & = \frac{a_{n-1}}{(n+2)(n+1)}. \end{aligned}$$

Let's plug in values of n to see what pattern emerges.

$$\begin{array}{lll} a_3 = \dfrac{a_0}{(3)(2)} & a_4 = \dfrac{a_1}{(4)(3)} & a_5 = \dfrac{a_2}{(5)(4)} = 0 \\ a_6 = \dfrac{a_3}{(6)(5)} & a_7 = \dfrac{a_4}{(7)(6)} & a_8 = \dfrac{a_5}{(8)(7)} \\ \quad = \dfrac{a_0}{(6)(5)(3)(2)} & \quad = \dfrac{a_1}{(7)(8)(4)(3)} & \quad = 0 \\ \quad \vdots & \quad \vdots & \quad \vdots \\ a_{3k} = \dfrac{a_0}{(2)(3)(5)(6)\cdots(3k-1)(3k)} & a_{3k+1} = \dfrac{a_1}{(3)(4)(7)(8)\cdots(3k)(3k+1)} & a_{3k+2} = 0 \end{array}$$

where $k = 1, 2, \ldots$ Now, we put this together.

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_3 x^3 + a_4 x^4 + a_6 x^6 + \ldots \\ &= a_0 + a_1 x + \ldots + \frac{a_0 x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)} + \frac{a_1 x^{3k+1}}{(3)(4)(7)(8)\cdots(3k)(3k+1)} + \ldots \\ &= a_0 \left(1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)} \right) + a_1 \left(x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \right) \end{aligned}$$

□

Now, let's look at what happens if we use the same equation, but a different point.

Example 30.6. *Compute the first five terms in the series solution around $x = 1$ to*

$$y'' - xy = 0.$$

In this case, we assume our solution looks like

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n,$$

and the derivatives are

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}.$$

So the differential equation becomes

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x \sum_{n=0}^{\infty} a_n (x-1)^n = 0.$$

Unlike in the previous example, we can't just multiply the x into the series. We have to make it look like $x-1$, which we do by adding and subtracting 1 from it.

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - (x-1+1)\sum_{n=0}^{\infty} a_n(x-1)^n = 0$$
$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - (x-1)\sum_{n=0}^{\infty} a_n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$
$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=0}^{\infty} a_n(x-1)^{n+1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

Now, we need to shift the first series up by 2 and the second down by 1 to get the same exponent before we strip out any extra terms and combine the series.

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=1}^{\infty} a_{n-1}(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$
$$2a_2 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_{n-1} - a_n\right](x-1)^n = 0$$

Setting the coefficients equal to 0,

$$\begin{aligned} n=0: &\qquad 2a_2 - a_0 = 0 \\ n>0: &\qquad (n+2)(n+1)a_{n+2} - a_{n-1} - a_n = 0, \end{aligned}$$

and solving gives

$$\begin{aligned} n=0: &\qquad a_2 = \frac{a_0}{2} \\ n>0: &\qquad a_{n+2} = \frac{a_{n-1}+a_n}{(n+2)(n+1)}. \end{aligned}$$

In this case, it would actually be very difficult, (if not impossible), for us to get general series coefficients. So all we care about doing is getting the first five terms. In general, this will be the case, but the important part is that we can compute as many terms as we care to have, so we can approximate the solution as closely as we like. So we'll just plug in values of n until we have enough terms.

$$\begin{aligned}
n=0: &\qquad a_2 = \frac{a_0}{2}\\
n=1: &\qquad a_3 = \frac{a_0+a_1}{(3)(2)}\\
&\qquad\quad = \frac{a_0}{6}+\frac{a_1}{6}\\
n=2: &\qquad a_4 = \frac{a_1+a_2}{(4)(3)}\\
&\qquad\quad = \frac{a_1}{12}+\frac{a_0}{24}\\
n=3: &\qquad a_5 = \frac{a_2+a_3}{(5)(4)}\\
&\qquad\quad = \frac{a_0}{40}+\frac{a_0}{120}+\frac{a_1}{120}\\
&\qquad\quad = \frac{a_0}{30}+\frac{a_1}{120}
\end{aligned}$$

Putting this together, our solution starts out as

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n(x-1)^n\\
&= a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + a_5(x-1)^5 + a_6(x-1)^6 + \dots\\
&= a_0 + a_1(x-1) + \frac{a_0}{2}(x-1)^2 + \left(\frac{a_0}{6}+\frac{a_1}{6}\right)(x-1)^3 + \left(\frac{a_0}{24}+\frac{a_1}{12}\right)(x-1)^4\\
&\qquad + \left(\frac{a_0}{30}+\frac{a_1}{120}\right)(x-1)^5 + \dots\\
&= a_0\left(1+\frac{(x-1)^2}{2}+\frac{(x-1)^3}{6}+\frac{(x-1)^4}{24}+\frac{(x-1)^5}{30}+\dots\right)\\
&\qquad + a_1\left(x+\frac{(x-1)^3}{6}+\frac{(x-1)^4}{12}+\frac{(x-1)^5}{120}+\dots\right).
\end{aligned}$$

□