LECTURE 2

Separable Equations

The most straightforward first order equations to solve are *separable equations*, which have the form

$$\boxed{\frac{dy}{dx} = f(y)g(x)} \tag{2.1}$$

It's critical that a separable equation has this form. If you can't write the equation as the product of a function only depending on y and a function only depending on x, the equation is *not* separable.

1. Solution Method

How do we solve them? The name should be revealing: the first step is to separate variables.

$$\frac{1}{f(y)}dy = g(x)dx$$
$$\int \frac{1}{f(y)}\,dy = \int g(x)\,dx$$
$$F(y) = G(x) + c$$

This gives an implicit solution for $y(x)$ (assuming we were able to compute both the integrals, which is not a given, as you're aware of from calculus), which we may be able to solve for explicitly, but in general we shouldn't expect to. At this point, if we have an initial condition we can plug it in to get the particular solution.

Remark. It is crucial not to forget the constant of integration in this solution process. In the form above, I've combined the two constants. Strictly speaking, after integration we would get

$$F(y) + c_1 = G(x) + c_2,$$

but if we just subtract c_1 from both sides we don't have to worry about writing down that extra step, since we just end up with some constant on one side anyway.

There's one other point to keep in mind: division by zero is never ok. This brings up a slight complication to our solution method, since we won't be accounting for those cases (if they occur in a given problem). The fix is fairly easy, though. If $f(y_0) = 0$, then $\frac{dy}{dx}(y_0, x) = 0$ for all x; hence if $y(x_0) = y_0$ for any x_0, then y is the constant function $y(x) = y_0$. Such a solution is called an *equilibrium solution*. Hence we have several solutions: $y(x) = y_k$ for any y_k satisfying $f(y_k) = 0$ and the general solution given by $F(y) = G(x) + c$ as given above. When just asked to "solve" the differential equation, you ought to take note of all of them; when looking for a particular solution, just see whether the initial y_0 is a zero of $f(y)$ and go from there.

2. Examples

Example 2.1 (Newton's Law of Cooling). Earlier, we saw Newton's Law of Cooling and wrote it as the differential equation

$$\frac{dB}{dt} = \kappa(E - B).$$

This is definitely separable: using our earlier notation, $f(B) = E - B$ (since E is a constant) and $g(t) = \kappa$. Now suppose we have some initial condition $B(0) = B_0$. We first note that if $B_0 = E$, this is an equilibrium solution, and in fact it is the only one. This makes sense physically, since if the body is at the same temperature as the environment, surely no heat transfer will occur. Let's find the non-equilibrium solutions:

$$\begin{aligned} \int \frac{dB}{E - B} &= \int \kappa \, dt \\ -\ln\|E - B\| &= \kappa t + c \\ E - B &= e^{-\kappa t + c} \\ &= Ae^{-\kappa t} \\ B(t) &= E - Ae^{-\kappa t}. \end{aligned}$$

In this case we were able to solve for $B(t)$ explicitly. Now if we use our initial condition:

$$\begin{aligned} B_0 &= E - A \\ A &= E - B_0 \end{aligned}$$

$$\boxed{B(t) = E - \frac{E - B_0}{e^{\kappa t}}}$$

□

Example 2.2.

$$\frac{dy}{dx} = 6y^2 x \qquad y(1) = \frac{1}{3}.$$

This is certainly in the form of Equation 2.1 with $f(y) = y^2$ and $g(x) = 6x$. First we look for the roots of $f(y)$, which tells us that the only equilibrium solution is $y(x) = 0$, but this isn't the particular solution we're looking for as the initial condition doesn't match up.

$$\begin{aligned} \int \frac{dy}{y^2} &= \int 6x \, dx \\ -\frac{1}{y} &= 3x^2 + c \\ -\frac{1}{\frac{1}{3}} &= 3(1^2) + c \Rightarrow c = -6 \\ -\frac{1}{y} &= 3x^2 - 6. \end{aligned}$$

$$\boxed{y(x) = \frac{1}{6 - 3x^2}}$$

What is the interval of validity for this solution? The only problems occur when $6 - 3x^2 = 0$, or when $x = \pm\sqrt{2}$. This solution, then, has three possible intervals of validity: $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, and $(\sqrt{2}, \infty)$. We want to choose the one containing our initial value of x, which in this case is $x = 1$, and this means that the interval of validity for the solution is $-\sqrt{2} < x < \sqrt{2}$. □

Exercise. In Example 2.2, verify that an initial condition of

$$y(-3) = -\frac{1}{21}$$

yields the same solution with an interval of validity of $(-\infty, -\sqrt{2})$. How about

$$y(2) = -\frac{1}{6}?$$

Example 2.3.

$$y' = \frac{3x^2 + 2x - 4}{2y - 2} \qquad y(1) = 3$$

There are no equilibrium solutions here, since there is no value of y that makes $\frac{1}{2y-2} = 0$, so:

$$\int 2y - 2\,dy = \int 3x^2 + 2x - 4\,dx$$
$$y^2 - 2y = x^3 + x^2 - 4x + c.$$

In a situation like this, it's convenient to compute the constant of integration before going on:

$$3^2 - 6 = 1 + 1 - 4 + c$$
$$3 = -2 + c \Rightarrow c = 5.$$

Now, we want to solve for $y(x)$ explicitly. We can do this either by completing the square or by using the quadratic formula:

$$y^2 - 2y + 1 = x^3 + x^2 - 4x + 5 + 1$$
$$(y-1)^2 = x^3 + x^2 - 4x + 6$$
$$y(x) = 1 \pm \sqrt{x^3 + x^2 - 4x + 6}.$$

This is two solutions; we need to choose the appropriate one. To do that, we'll use the initial condition, which is that $y(1) = 3$. Clearly, we will need to choose the "plus" solution rather than the "minus" one, since the square root is always positive and $3 > 1$.

$$\boxed{y(x) = 1 + \sqrt{x^3 + x^2 - 4x + 6}}$$

The issue of computing the interval of validity is a bit more complicated, since it would require us to find the roots of $x^3 + x^2 - 4x + 6$, which unfortunately are not integers. But the principle is straightforward enough: we need to avoid those values of x such that $x^3 + x^2 - 4x + 6 < 0$. So we find the appropriate interval around $x = 1$ and that's the interval of validity. □

Example 2.4.

$$\frac{dy}{dx} = \frac{xy^3}{1 + x^2} \qquad y(0) = 1$$

We only have one equilibrium solution, namely, $y(x) = 0$, which isn't our case, so let's separate:

$$\int \frac{dy}{y^3} = \int \frac{x}{1 + x^2}\,dx$$
$$-\frac{1}{2y^2} = \frac{1}{2}\ln\left(1 + x^2\right) + c$$

Use the inital condition:

$$-\frac{1}{2} = c.$$

So:

$$y^2 = \frac{1}{1 - \ln(1 + x^2)}$$

$$\boxed{y(x) = \frac{1}{\sqrt{1 - \ln(1 + x^2)}}}$$

Now all that's left is to determine the interval of validity. We know that we can't divide by zero, nor can the quantity inside the square root be negative. Acordingly, our requirement is

$$\ln\left(1+x^2\right) < 1$$
$$x^2 < e-1$$

So the desired interval of validity is $-\sqrt{e-1} < x < \sqrt{e-1}$. □

Example 2.5.

$$x^2\frac{dy}{dx} = y-1 \qquad y(0)=1$$

Putting this into separable form, we get

$$\frac{dy}{dx} = \frac{y-1}{x^2}.$$

There's one equilibrium solution: $y=1$. And our initial condition is that $y(0)=1$, so in this case, the desired solution is just the constant function $y(x)=1$. □

Example 2.6.

$$\frac{dy}{dt} = e^{y-t}\sec(y)\left(1+t^2\right) \qquad y(0)=0$$

Before we can separate this, we first need to rewrite it slightly, so it's in separable form, and we'll see we have no equilibrium solutions, so we need to continue:

$$\frac{dy}{dt} = \frac{e^y e^{-t}}{\cos(y)}\left(1+t^2\right)$$
$$\int e^{-y}\cos(y)\,dy = e^{-t}\left(1+t^2\right)\,dt.$$

Both sides require integration by parts, and after doing that we'll get an implicit solution.

$$\frac{e^{-y}}{2}\left(\sin(y)-\cos(y)\right) = -e^{-t}\left(t^2+2t+3\right)+c$$

Now we'll use the initial condition.

$$\frac{1}{2}(-1) = -3+c \Rightarrow c = \frac{5}{2}$$

$$\boxed{\frac{e^{-y}}{2}\left(\sin(y)-\cos(y)\right) = -e^{-t}\left(t^2+2t+3\right)+\frac{5}{2}}$$

In this case, we won't be able to find an explicit solution, so we have to leave it in implicit form. It's generally very difficult to find the interval of validity when we have an implicit solution, so we won't bother for this problem. □