LECTURE 9

Second Order Linear Differential Equations

1. Basic Concepts

We've seen one particular example of a second order linear differential equation before: Newton's Second Law, which can be written as a second order equation for position $s(t)$ as

$$\frac{d^2s}{dt^2} = mF(t, s', s).$$

One of the goals of this section of the course will be to derive differential equations for spring-mass systems that build on this differential equation. Before we can do that, however, we'll need to develop the necessary mathematical tools.

One of the most basic second order differential equations is $y'' = -y$. By inspection, we might notice that this has two obvious nonzero solutions: $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$. But what about $y_3(t) = 9\cos(t) - 2\sin(t)$? This is also a solution. So does any $y(t) = c_1 \cos(t) + c_2 \sin(t)$, where c_1 and c_2 are constants. In fact, every solution to this differential equation is of this form.

Example 9.1. *Find all of the solutions to $y'' = 9y$.*

We need to think of a function whose second derivative is 9 times the original function. What function comes back to itself (without a sign change) after two derivatives? We might think of the exponential function, and checking exponents indicates that the following two functions are solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$. In fact, so are any functions of the solution $y(t) = c_1e^{3t} + c_2e^{-3t}$. □

Exercise. Check that $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$ are in fact solutions to $y'' = 9y$.

The general form of a second order linear differential equation is

$$p(t)y'' + q(t)y' + r(t)y = g(t).$$

We call the equation *homogeneous* if $g(t) = 0$ and *nonhomogeneous* if $g(t) \neq 0$.

Why is that? The first example discussed, $y'' = -y$, is indicative: we can see that there are two "different" solutions, and any linear combination of them is also a solution. This is in general true, and is important enough to have a name.

Theorem 9.1 (Principle of Superposition). *If $y_1(t)$ and $y_2(t)$ are solutions to a second order linear homogeneous differential equation, then so is any linear combination*

$$y(t) = c_1y_1(t) + c_2y_2(t).$$

This follows from homogeneity and the fact that the derivative is linear (so we can pull out the constants and split up derivatives of $y(t)$ into the derivatives of y_1 and y_2). So we know that if we have two solutions, we can find many more by taking linear combinations, but ultimately we would like to be able to write down a general solution to the differential equation, so that with some initial conditions we could uniquely solve an initial value problem. The basic idea is that we want to find two "different" solutions $y_1(t)$ and $y_2(t)$, so that the general solution to the differential equation is $y(t) = c_1y_1(t) + c_2y_2(t)$. What does "different" mean, exactly? We'll hold off on answering that question for a few lectures. It's not too hard to get a feel for when two solutions are "different" enough; the main point is that they shouldn't be constant multiples.

Now, let's think back to the example of $y'' = -y$. We found two "different" solutions, $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ and saw that any solution can be written as a linear combination of these two solutions, $y(t) = c_1 \cos(t) + c_2 \sin(t)$. We have two constants, so we'll need two initial conditions to find a particular solution. We'll generally give these conditions by specifying the values of y and y' at a particular t_0. So, for us, a second order linear homogeneous initial value problem might look like

$$p(t)y'' + q(t)y' + r(t)y = 0 \qquad y'(t_0) = y_0', y(t_0) = y_0.$$

EXAMPLE 9.2. *Find the particular solution to the initial value problem*

$$y'' + y = 0 \qquad y(0) = 2, y'(0) = -1.$$

We're taking for granted for now that the general solution to this equation is

$$y(t) = c_1 \cos(t) + c_2 \sin(t).$$

To apply the initial conditions, we'll need to know the derivative:

$$y'(t) = -c_1 \sin(t) + c_2 \cos(t).$$

Plugging in the initial conditions yields

$$\begin{aligned} 2 &= c_1 \\ -1 &= c_2, \end{aligned}$$

so the particular solution is

$$y(t) = 2\cos(t) - \sin(t).$$

□

Sometimes applying initial conditions will mean we'll have to solve a system of equations; other times it's as easy as the previous example.

2. Homogeneous Equations With Constant Coefficients

To begin with, we'll consider the easiest class of second order linear homogeneous equations, where the coefficient functions $p(t), q(t)$, and $r(t)$ are constants. In other words, the equations we'll look at have the form

$$\boxed{ay'' + by' + cy = 0.} \tag{9.1}$$

How do we begin finding solutions to this equation? Somehow, the solution $y(t)$ is linked to its first and second derivatives just by addition and multiplication by constants. If we think back to calculus, we can find a function that is linked to its derivative by a multiplicative constant: $y(t) = e^{rt}$. To see if this is on the right track, let's plug it into the differential equation 9.1 and see what we get. To do this, we'll need the derivatives $y'(t) = re^{rt}$ and $y''(t) = r^2 e^{rt}$.

$$\begin{aligned} a\left(r^2 e^{rt}\right) + b\left(re^{rt}\right) + ce^{rt} &= 0 \\ e^{rt}\left(ar^2 + br + c\right) &= 0 \end{aligned}$$

What can we conclude? If $y(t) = e^{rt}$ is a solution to the differential equation, then $e^{rt}(ar^2+br+c) = 0$. Since exponentials are never zero, we're left with the condition that $y(t) = e^{rt}$ will solve the differential equation as long as r is a solution to

$$ar^2 + br + c = 0.$$

This equation is called the *characteristic equation* for 9.1.

Thus, to find solutions to a linear second order constant coefficient equation, we begin by writing down the characteristic equation. Then we find the roots r_1 and r_2 (note that these don't necessarily have to be distinct, or even real). At this point, we'll have some solutions to the differential equation:

$$y_1(t) = e^{r_1 t} \qquad y_2(t) = e^{r_2 t}.$$

Of course, it's also possible these are the same, since we might have a repeated root. As a result, we'll put off mentioning whether or not these two solutions are "different" enough to form the general solution to Equation 9.1. In fact, we have three cases, and in each case this question will be addressed differently.

Let's look at an example.

EXAMPLE 9.3. *Find two solutions to the differential equation $y'' - 9y = 0$.*

Notice that this is the same equation as in Example 9.1. The characteristic equation is $r^2 - 9 = 0$, and this has roots $r = \pm 3$. So we have two solutions $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$, which agree with our earlier guesses. □

So what are the three cases that were mentioned earlier? They're the same as the three possibilities for types of roots of quadratic equations:

(1) Real, distinct roots $r_1 \neq r_2$.
(2) Complex roots $r_1, r_2 = \alpha \pm \beta i$.
(3) A repeated real root $r_1 = r_2 = r$.

We'll look at each case more closely in the lectures to come.