

LECTURE 34

Repeated Eigenvalues

The final case we need to consider involves repeated eigenvalues. In the previous two cases, we had distinct eigenvalues, which, as Theorem 31.3 a few lectures ago told us, have linearly independent eigenvectors. Thus, all we had to do was to calculate those eigenvectors and write down solutions of the form

$$\mathbf{x}_i(t) = \boldsymbol{\eta}^{(i)} e^{\lambda_i t}.$$

In the real case, the linear independence of the eigenvectors and the differing eigenvalues assured us that these two solutions formed a fundamental set, and so our general solution was just a linear combination of the two. In the complex case, we could have done something similar, but instead we were able to form our real-valued general solution out of a single complex-valued solution.

When we have an eigenvalue of multiplicity 2, however, Theorem 31.3 tells us that we could have either one or two eigenvectors up to linear independence. If we have two, that's fine; if we have one, we'll clearly have more work to do. Let's consider each of these situations.

1. A Complete Eigenvalue

We call a repeated eigenvalue *complete* if it has two distinct (as in, linearly independent) eigenvectors. This case is actually quite simple. Suppose our repeated eigenvalue λ has two linearly independent eigenvectors $\boldsymbol{\eta}^{(1)}$ and $\boldsymbol{\eta}^{(2)}$. Then we can proceed as before, and our general solution is

$$\begin{aligned}\mathbf{x}(t) &= c_1 e^{\lambda t} \boldsymbol{\eta}^{(1)} + c_2 e^{\lambda t} \boldsymbol{\eta}^{(2)} \\ &= e^{\lambda t} (c_1 \boldsymbol{\eta}^{(1)} + c_2 \boldsymbol{\eta}^{(2)}).\end{aligned}$$

It's a basic fact from linear algebra that given two linearly independent vectors such as $\boldsymbol{\eta}^{(1)}$ and $\boldsymbol{\eta}^{(2)}$, we can form any other two-dimensional vector out of a linear combination of these two. So what we have is a situation where any vector function of the form $\mathbf{x}(t) = e^{\lambda t} \boldsymbol{\eta}$ is a solution. As discussed earlier, this can only happen if $\boldsymbol{\eta}$ is an eigenvector of the coefficient matrix with eigenvalue λ . The conclusion, then, is that if λ has two linearly independent eigenvectors, *every* vector is an eigenvector.

This can only happen if the coefficient matrix A is a scalar multiple of the identity matrix, as we need

$$A\boldsymbol{\eta} = \lambda\boldsymbol{\eta} = \lambda I\boldsymbol{\eta}$$

for every vector $\boldsymbol{\eta}$. Thus, this case only arises when

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

or when the original system of equations is

$$\begin{aligned}x_1 &= \lambda x_1 \\ x_2 &= \lambda x_2.\end{aligned}$$

What does the phase portrait look like in this case? Since every vector is an eigenvector, every trajectory that isn't the constant solution at the origin is an eigensolution and hence a straight line. We call such the equilibrium solution in this case a *star node* and its stability is determined

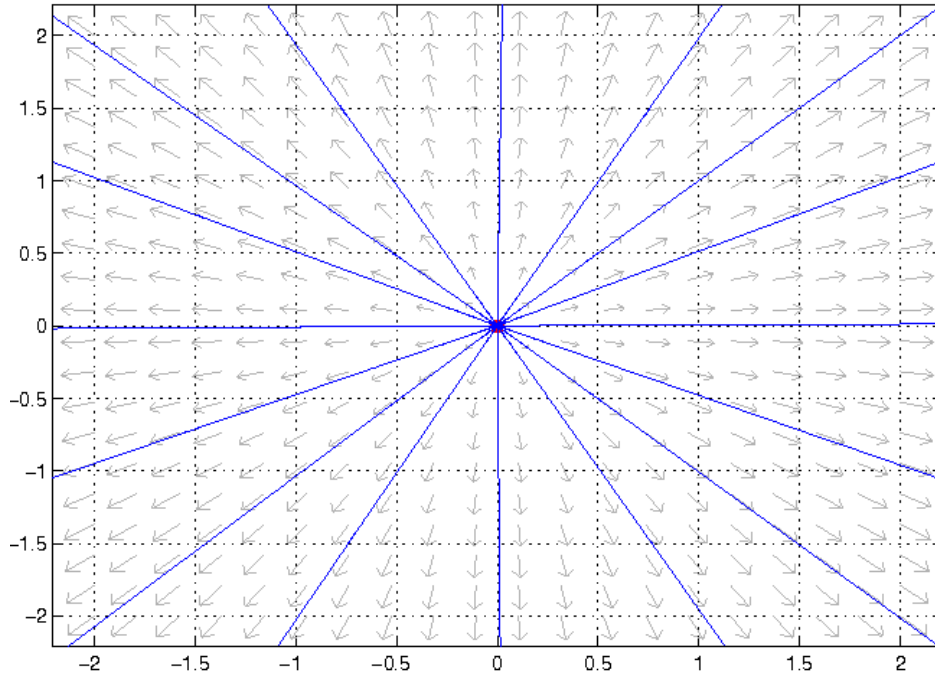


FIGURE 34.1. An example of the phase portrait for a star node.

by the sign of λ : if $\lambda > 0$, all of the eigensolutions grow away from the origin and the origin is unstable, but if $\lambda < 0$, every solution decays to the origin and the origin is asymptotically stable. We obtain something like Figure 34.1.

This is a fairly degenerate (and isolated) situation that won't come up in further discussion, but it's important to keep in mind that this can happen.

2. A Defective Eigenvalue

The other possibility is that λ only has a single eigenvector η up to linear independence. In this case, we have a bit of a problem: to form a general solution we need two linearly independent solutions, but we only have one, namely

$$\mathbf{x}_1(t) = e^{\lambda t}\eta.$$

In this case, we say that λ is *defective* or *incomplete*. What can we do?

We had a similar problem in the second order linear case. When we ran into this situation there, we were able to work around it by multiplying the solution by a t . Let's try to mimic that and see if

$$\mathbf{x}(t) = te^{\lambda t}\eta$$

is a solution.

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ \lambda te^{\lambda t}\eta + \lambda\eta te^{\lambda t} &= A\eta te^{\lambda t} \end{aligned}$$

Matching coefficients, in order for this guess to be a solution, we require

$$\begin{aligned} \eta e^{\lambda t} = \mathbf{0} &\Rightarrow \eta = \mathbf{0} \\ \lambda\eta e^{\lambda t} = A\eta e^{\lambda t} &\Rightarrow (A - \lambda I)\eta = \mathbf{0}. \end{aligned}$$

Thus we need $\boldsymbol{\eta}$ to be an eigenvector of A , which we knew it was, but we also need $\boldsymbol{\eta} = \mathbf{0}$, which is right out. So this guess doesn't work and we'll need another approach.

What was the problem with our earlier calculation? We ended up with a term that didn't have a t , but rather just an exponential in it, and this term caused us to require $\boldsymbol{\eta} = \mathbf{0}$. A possible fix might be to add in another term to our guess that only involves an exponential and some other vector $\boldsymbol{\rho}$. Let's guess that the form of the solution might be

$$\mathbf{x}(t) = te^{\lambda t}\boldsymbol{\eta} + e^{\lambda t}\boldsymbol{\rho}$$

and see what conditions on $\boldsymbol{\rho}$ we can derive.

$$\begin{aligned}\mathbf{x}' &= A\mathbf{x} \\ \lambda\boldsymbol{\eta}te^{\lambda t} + \boldsymbol{\eta}e^{\lambda t} + \lambda\boldsymbol{\rho}e^{\lambda t} &= A\left(\boldsymbol{\eta}te^{\lambda t} + \boldsymbol{\rho}e^{\lambda t}\right) \\ (\boldsymbol{\eta} + \lambda\boldsymbol{\rho})e^{\lambda t} + \boldsymbol{\eta}te^{\lambda t} &= A\boldsymbol{\eta}te^{\lambda t} + A\boldsymbol{\rho}e^{\lambda t}\end{aligned}$$

Thus, setting coefficients equal again, we have

$$\begin{aligned}A\boldsymbol{\eta} = \lambda\boldsymbol{\eta} &\Rightarrow (A - \lambda I)\boldsymbol{\eta} = \mathbf{0} \\ \boldsymbol{\eta} + \lambda\boldsymbol{\rho} = A\boldsymbol{\rho} &\Rightarrow (A - \lambda I)\boldsymbol{\rho} = \boldsymbol{\eta}.\end{aligned}$$

This first condition only tells us that $\boldsymbol{\eta}$ is an eigenvector of A , which we already knew. But the second condition is more useful. It tells us that if $(A - \lambda I)\boldsymbol{\rho} = \boldsymbol{\eta}$, then

$$\mathbf{x}(t) = \boldsymbol{\eta}te^{\lambda t} + \boldsymbol{\rho}e^{\lambda t}$$

will be a solution to the differential equation.

A vector $\boldsymbol{\rho}$ satisfying

$$(A - \lambda I)\boldsymbol{\rho} = \boldsymbol{\eta}$$

is called a *generalized eigenvector*, because while $(A - \lambda I)\boldsymbol{\rho} \neq \mathbf{0}$, it's not hard to verify that

$$(A - \lambda I)^2\boldsymbol{\rho} = \mathbf{0}.$$

So as long as we can produce a generalized eigenvector $\boldsymbol{\rho}$, this formula will give us a second solution and we can form a general solution.

EXAMPLE 34.1. Find the general solution to the following system.

$$\mathbf{x}' = \begin{pmatrix} -6 & -5 \\ 5 & 4 \end{pmatrix} \mathbf{x}$$

We begin by finding the eigenvalues.

$$\begin{aligned}0 = \det(A - \lambda I) &= \begin{vmatrix} -6 - \lambda & -5 \\ 5 & 4 - \lambda \end{vmatrix} \\ &= \lambda^2 + 2\lambda + 1 \\ &= (\lambda + 1)^2\end{aligned}$$

So we have a repeated eigenvalue of $\lambda = -1$. Due to the form of the matrix, we can also figure out from our previous discussion that λ will only have one eigenvector up to linear independence. Let's calculate it by solving $(A + I)\boldsymbol{\eta} = \mathbf{0}$.

$$\begin{pmatrix} -5 & -5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So the system of equations we want to solve is

$$\begin{aligned} -5\eta_1 - 5\eta_2 &= 0 \\ 5\eta_1 + 5\eta_2 &= 0. \end{aligned}$$

This is solved by anything of the form $\eta_1 = -\eta_2$. So if we choose $\eta_2 = 1$, we get an eigenvector of

$$\boldsymbol{\eta} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

This isn't enough, though. We also need to find a generalized eigenvector $\boldsymbol{\rho}$. So we need to solve $(A + I)\boldsymbol{\rho} = \boldsymbol{\eta}$, or

$$\begin{pmatrix} -5 & -5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \rho_1 = \frac{1}{5} - \rho_2.$$

So our generalized eigenvector has the form

$$\boldsymbol{\rho} = \begin{pmatrix} \frac{1}{5} - \rho_2 \\ \rho_2 \end{pmatrix}.$$

Here, we can choose any value of ρ_2 we want. Choosing $\rho_2 = 0$ gives us

$$\boldsymbol{\rho} = \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix}.$$

Thus our general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{\lambda t} \boldsymbol{\eta} + c_2 \left(t e^{\lambda t} \boldsymbol{\eta} + e^{\lambda t} \boldsymbol{\rho} \right) \\ &= c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left(t e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix} \right). \end{aligned}$$

□

EXAMPLE 34.2. *Sketch the phase portrait for the system in Example 34.1.*

We begin by drawing our eigensolution. Note that in this case, we only have one, unlike the case where we had a (nondegenerate) node. The eigensolution is the straight line in the direction $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, as indicated in Figure 34.2. As the eigenvalue is negative, this solution will decay towards the origin.

But what happens to other trajectories? First, let's just consider the general solution

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left(t e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix} \right)$$

with $c_2 \neq 0$. All three terms have the same exponential, but as $t \rightarrow \pm\infty$ the $t e^{-t}$ term will have a larger magnitude than the other two. Thus, in both forward and backward time, we would get that the trajectories will become parallel to the single eigensolution, as in Figure 34.2.

Now, since $\lambda < 0$ and exponentials decay faster than polynomials grow, we can see that as $t \rightarrow \infty$, every solution will decay to the origin. So the origin will be asymptotically stable. We also call the origin a *degenerate node* in this case, as it behaves analogously to how a node behaves but only has a single eigensolution.

Here's one way to think about how a degenerate node should behave. Consider a node with two close eigenvalues. Then try to imagine what happens to 32.3 as we bring the eigenvalues together. The eigensolutions will collapse together, but the non-eigensolution trajectories would keep their asymptotic behavior with regard to this collapsed eigensolution.

Notice that, as illustrated in Figure 34.2, we end up with a large degree of rotation of the solution. The solution has to turn around to be able to be asymptotic to the solution in both

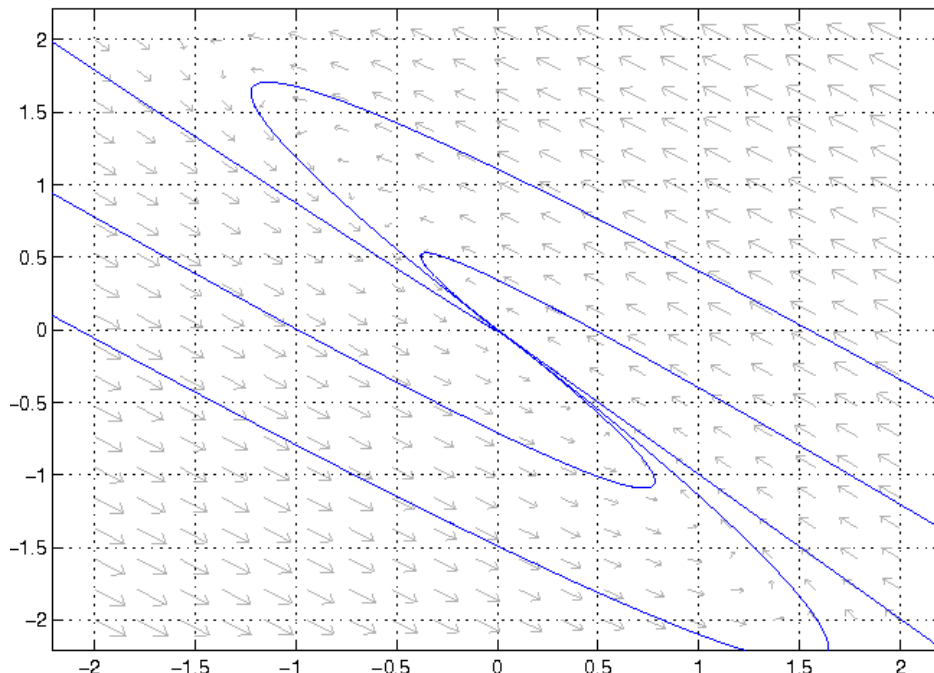


FIGURE 34.2. Phase portrait for the asymptotically stable degenerate node in Example 34.1.

forward and backward time. This is because, in another manner of thinking, degenerate nodes are the borderline case between nodes and spirals. Suppose our characteristic equation is

$$0 = \lambda^2 + b\lambda + c.$$

The eigenvalues are then, by the quadratic formula,

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{a}.$$

The discriminant of this equation is positive in the node case and negative in the spiral/center cases. We get degenerate nodes when the solutions transition between these two cases and the discriminant becomes zero. So a way of thinking about the degenerate node is that the solutions are trying to wind around in a spiral, but they don't quite make it due to the lack of complexity of the eigenvalue.

But how do we know the direction of rotation? We do the same thing we did in the spiral case: compute the tangent vector at a point or two. That, combined with our knowledge of the stability of the origin, will tell us how the non-eigensolutions must turn.

Let's start by considering the point $(1, 0)$. At this point,

$$\mathbf{x}' = \begin{pmatrix} -6 & -5 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}.$$

As we know that our solution is asymptotically stable, the tangent vector can only point up and to the left if the solutions rotate counterclockwise as they start to approach the origin. \square

EXAMPLE 34.3. Find the general solution to the following system.

$$\mathbf{x}' = \begin{pmatrix} 12 & 4 \\ -16 & -4 \end{pmatrix} \mathbf{x}$$

We begin by finding the eigenvalues.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 12 - \lambda & 4 \\ -16 & -4 - \lambda \end{vmatrix} \\ &= \lambda^2 - 8\lambda + 16 \\ &= (\lambda - 4)^2 \end{aligned}$$

So we have a repeated eigenvalue of $\lambda = 4$. Let's calculate an eigenvalue by solving $(A - 4I)\boldsymbol{\eta} = \mathbf{0}$.

$$\begin{pmatrix} 8 & 4 \\ -16 & -8 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So the system of equations we want to solve is

$$\begin{aligned} 8\eta_1 + 4\eta_2 &= 0 \\ -16\eta_1 - 8\eta_2 &= 0. \end{aligned}$$

This is solved by anything of the form $\eta_2 = -2\eta_1$. So if we choose $\eta_1 = 1$, we get an eigenvector of

$$\boldsymbol{\eta} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Now let's find a generalized eigenvector $\boldsymbol{\rho}$. We need to solve $(A - 4I)\boldsymbol{\rho} = \boldsymbol{\eta}$, or

$$\begin{pmatrix} 8 & 4 \\ -16 & -8 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \Rightarrow \quad \rho_2 = \frac{1}{4} - 2\rho_1.$$

So our generalized eigenvector has the form

$$\boldsymbol{\rho} = \begin{pmatrix} \rho_1 \\ \frac{1}{4} - 2\rho_1 \end{pmatrix}.$$

Choosing $\rho_1 = 0$ gives us

$$\boldsymbol{\rho} = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}.$$

Thus our general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{\lambda t} \boldsymbol{\eta} + c_2 \left(t e^{\lambda t} \boldsymbol{\eta} + e^{\lambda t} \boldsymbol{\rho} \right) \\ &= c_1 e^{4t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \left(t e^{4t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} \right). \end{aligned}$$

□

EXAMPLE 34.4. *Sketch the phase portrait of the system in Example 34.3.*

Everything here is completely analogous to the previous example's phase portrait. We sketch the eigensolution, and note that it will grow away from the origin as $t \rightarrow \infty$, so the origin will in this case be an unstable degenerate node.

Typical trajectories will once again come out of the origin parallel to the eigensolution and rotate around to be parallel to them again, and all we would need to do is to calculate the direction of rotation by computing the tangent vector at a point or two. At $(1, 0)$, we would get

$$\mathbf{x}' = \begin{pmatrix} 16 \\ -12 \end{pmatrix},$$

which can only happen given that the solutions are growing if the direction of rotation is clockwise. Thus we get Figure 34.3. □

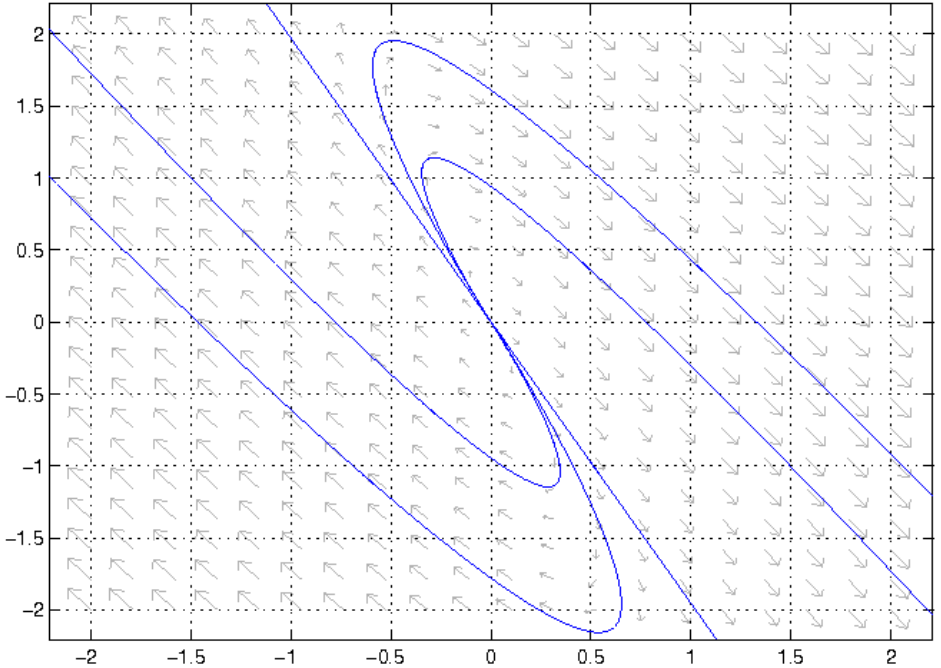


FIGURE 34.3. Phase portrait of the unstable degenerate node in Example 34.3.