LECTURE 12

Reduction of Order

1. Repeated Roots, Part Deux

Last class, we saw that the differential equation

$$ay'' + by' + cy = 0$$

has general solution

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}$$

when the characteristic equation $ar^2 + br + c$ has a single repeated root r. Now, let's look at some examples.

Example 12.1. *Solve the IVP*

$$y'' - 4y' + 4y = 0 \qquad y(0) = -1, y'(0) = 6.$$

The characteristic equation is

$$\begin{aligned} r^2 - 4r + 4 &= 0 \\ (r-2)^2 &= 0 \end{aligned}$$

so we see that we have a repeated root $r = 2$. The general solution and its derivative are

$$\begin{aligned} y(t) &= c_1 e^{2t} + c_2 t e^{2t} \\ y'(t) &= 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} \end{aligned}$$

and plugging in initial conditions yields

$$\begin{aligned} -1 &= c_1 \\ 6 &= 2c_1 + c_2 \end{aligned}$$

so we have $c_1 = -1$ and $c_2 = 8$. The particular solution is

$$y(t) = -e^{2t} + 6te^{2t}.$$

□

Example 12.2. *Solve the IVP*

$$16y'' + 40y' + 25y = 0 \qquad y(0) = -1, y'(0) = 2.$$

The characteristic equation is

$$\begin{aligned} 16r^2 + 40r + 25 &= 0 \\ (4r+5) &= 0 \end{aligned}$$

and so we conclude that we have a repeated root $r = -\frac{5}{4}$ and the general solution and its derivative are

$$\begin{aligned} y(t) &= c_1 e^{-\frac{5}{4}t} + c_2 t e^{-\frac{5}{4}t} \\ y'(t) &= -\frac{5}{4} c_1 e^{-\frac{5}{4}t} + c_2 e^{-\frac{5}{4}t} - \frac{5}{4} c_2 t e^{-\frac{5}{4}t}. \end{aligned}$$

Plugging in the initial conditions yields

$$\begin{aligned} -1 &= c_1 \\ 2 &= -\frac{5}{4}c_1 + c_2 \end{aligned}$$

so $c_1 = -1$ and $c_2 = \frac{3}{4}$. The particular solution is

$$y(t) = -e^{-\frac{5}{4}t} + \frac{3}{4}te^{-\frac{5}{4}t}.$$

□

2. Reduction of Order

We've just spent a few classes discussing second order linear homogeneous equations with constant coefficients, *i.e.*, equations with the form

$$ay'' + by' + cy = 0.$$

Now, before we briefly discuss the theory of second order linear homogeneous equations, let's consider what happens if our coefficients aren't constant. In other words, we're looking at equations of the form

$$p(t)y'' + q(t)y' + r(t)y = 0.$$

In general, finding solutions to these equations isn't easy, but if we know (or can guess at) a solution, we can use the technique we used in the repeated roots case to find a second solution that is "different" enough from our original solution to give us a general solution. This method is known as *reduction of order.*

Let's look at an example to begin with.

Example 12.3. *Find the general solution to*

$$2t^2y'' + ty' - 3y \tag{12.1}$$

given that $y_1(t) = t^{-1}$ is a solution.

Let's think back to the repeated roots case we discussed last class. We knew we had a solution y_1 and needed to find a distinct solution. What did we do? We asked which nonconstant functions $v(t)$ make $y_2(t) = v(t)y_1(t)$ also a solution. Let's do the same thing here.

Set $y_2(t) = v(t)y_1(t)$. Then y_2 and its derivatives are

$$\begin{aligned} y_2 &= vt^{-1} \\ y_2' &= v't^{-1} - vt^{-2} \\ y_2'' &= v''t^{-1} - v't^{-2} - v't^{-2} + 2vt^{-3} = v''t^{-1} - 2v't^{-2} + 2vt^{-3}. \end{aligned}$$

The next step is to plug into Equation 12.1 so we can solve for v:

$$\begin{aligned} 2t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + t\left(v't^{-1} - vt^{-2}\right) - 3vt^{-1} &= 0 \\ 2v''t - 4v' + 4vt^{-1} + v' - vt^{-1} - 3vt^{-1} &= 0 \\ 2tv'' - 3v' &= 0. \end{aligned}$$

Notice that the only terms left involve v'' and v', not v. This also happened in the repeated roots case (where the v' term also dropped out, but that in general won't happen). The v term should always disappear at this point, so this is a good spot to make sure that we've done our differentiation and algebra are correct. If there's a v term left, we haven't.

Now, we know that if y_2 is a solution, the function v must satisfy

$$2tv'' - 3v = 0.$$

But this is a second order linear homogeneous equation with nonconstant coefficients. All we've done is to replace our original one with this new one. So have we actually helped ourselves?

Yes, we have. Since there is no v term in the new differential equation, we can make the substitution $w(t) = v'(t)$ (which also gives $w'(t) = v''(t)$). By changing variables in this way, our equation becomes

$$2tw' - 3w = 0. \tag{12.2}$$

This is a first order linear equation, which we know how to solve. This is precisely why the method is called reduction of order: we're taking a second order equation and using a known solution to make it first order.

Alright, so let's go ahead and solve Equation 12.2. First, we need to put it in the correct form for integrating factors:

$$w' - \frac{3}{2t}w = 0.$$

So our integrating factor is

$$\mu(t) = e^{\int -\frac{3}{2t}\,dt} = e^{-\frac{3}{2}\ln(t)} = t^{-\frac{3}{2}}.$$

$$\begin{aligned}\left[t^{-\frac{3}{2}}w\right]' &= 0\\ t^{-\frac{3}{2}}w &= c\\ w(t) &= ct^{\frac{3}{2}}\end{aligned}$$

So we know what $w(t)$ must be to solve Equation 12.2. But to solve Equation 12.1, our original differential equation, we don't need $w(t)$, we need $v(t)$. Since $v'(t) = w(t)$, integrating w will give our v.

$$\begin{aligned}v(t) &= \int w(t)\,dt\\ &= \int ct^{\frac{3}{2}}\,dt\\ &= \frac{2}{5}ct^{\frac{5}{2}} + k\end{aligned}$$

Now, this is the most general form for $v(t)$: for any constants c, k, $y_2(t) = v(t)y_1(t)$ will solve Equation 12.1. So, just like in the repeated roots case, we can choose c and k to pick one particular such $v(t)$ that's nice and simple. The only constraint is that we can't choose $c = 0$, since then $v(t)$ would be a constant and y_1 and y_2 would be essentially the same. A natural choice is to take $c = \frac{5}{2}$ and $k = 0$. Then $v(t) = t^{\frac{5}{2}}$, so $y_2(t) = v(t)y_1(t) = t^{\frac{3}{2}}$, and the general solution to Equation 12.1 is

$$y(t) = c_1t^{-1} + c_2t^{\frac{3}{2}}.$$

□

Reduction of order is a very nice and powerful method for finding a second solution to a differential equation when we don't have any other method, but it does require us to start with a given (or guessed at) solution. Sometimes finding this first solution is very difficult, but once we have it, we can reduce the order to find the second solution and hence a general solution.

It's also important to be careful with these problems: sometimes the algebra gets a little nasty, and it's easy to make sloppy mistakes. It's important to make sure that the v term does actually drop out of the equation after we plug in the derivatives of y_2 and it's also important to check the second solution we obtain in case we made an algebraic error somewhere.

Let's do one more example.

Example 12.4. *Find the general solution to*

$$t^2y'' + 2ty' - 2y = 0$$

given that

$$y_1(t) = t$$

is a solution.

We start by setting $y_2(t) = v(t)y_1(t)$. So we have

$$\begin{aligned} y_2 &= tv \\ y_2' &= tv' + v \\ y_2'' &= tv'' + v' + v' = tv'' + 2v'. \end{aligned}$$

Next, we plug in and arrange terms.

$$\begin{aligned} t^2\left(tv'' + 2v'\right) + 2t\left(tv' + v\right) - 2tv &= 0 \\ t^3v'' + 2t^2v' + 2t^2v' + 2tv - 2tv &= 0 \\ t^3v'' + 4t^2v' &= 0. \end{aligned}$$

Notice that, as desired, the v term drops out. We make the change of variable $w(t) = v'(t)$ to obtain

$$t^3w' + 4t^2w = 0$$

, which has integrating factor $\mu(t) = t^4$.

$$\begin{aligned} \left[t^4w\right]' &= 0 \\ t^4w &= c \\ w(t) &= ct^{-4} \end{aligned}$$

So we have

$$\begin{aligned} v(t) &= \int w(t)\,dt \\ &= \int ct^{-4}\,dt \\ &= -\frac{c}{3}t^{-3} + k. \end{aligned}$$

A nice choice for the constants is $c = -3$ and $k = 0$, so we get $v(t) = t^-3$, which gives a second solution of $y_2(t) = v(t)y_1(t) = t^{-2}$. So our general solution is

$$y(t) = c_1t + c_2t^{-2}.$$

□