LECTURE 10

Distinct Roots of the Characteristic Equation

1. Real, Distinct Roots

Recall from last time that a second order linear homogeneous differential equation with constant coefficients

$$ay'' + by' + cy = 0 \tag{10.1}$$

is solved by $y(t) = e^{rt}$, where r solves the *characteristic equation*

$$ar^2 + br + c = 0. \tag{10.2}$$

So, when the characteristic equation has two distinct real roots $r_1 \neq r_2$, we get two solutions $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$. It will turn out that, in this case, $y_1(t)$ and $y_2(t)$ are "different" enough so that the general solution of Equation 10.1 is given by

$$\boxed{y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.}$$

Then, given initial conditions, we can solve for c_1 and c_2.

Example 10.1. *Solve the IVP*

$$y'' + 3y' - 18y = 0 \qquad y(0) = 0, y'(0) = -1.$$

The characteristic equation is

$$r^2 + 3r - 18 = 0$$
$$(r + 6)(r - 3) = 0$$

so the roots are $r_1 = -6$ and $r_2 = 3$. Thus the general solution and its derivative are

$$y(t) = c_1 e^{-6t} + c_2 e^{3t}$$
$$y'(t) = -6c_1 e^{-3t} + 3c_2 e^{3t}$$

and plugging in the initial conditions yields the system of equations

$$0 = c_1 + c_2$$
$$-1 = -6c_1 + 3c_2$$

which has solution $c_1 = \frac{1}{9}$ and $c_2 = -\frac{1}{9}$. Thus the particular solution is

$$y(t) = \frac{1}{9} e^{-6t} - \frac{1}{9} e^{3t}.$$

□

Example 10.2. *Solve the IVP*

$$y'' - 7y' + 10y = 0 \qquad y(0) = 3, y(0) = 2.$$

The characteristic equation is

$$r^2 - 7r + 10 = 0$$
$$(r - 5)(r - 2) = 0.$$

The roots are $r_1 = 5$ and $r_2 = 2$, so the general solution and its derivative are

$$y(t) = c_1 e^{5t} + c_2 e^{2t}$$
$$y'(t) = 5c_1 e^{5t} + 2c_2 e^{2t}.$$

Plugging in the initial conditions yields the system of equations

$$3 = c_1 + c_2$$
$$2 = 5c_1 + 2c_2$$

which has solution $c_1 = -\frac{4}{3}$, $c_2 = \frac{13}{3}$, and so the particular solution is

$$y(t) = -\frac{4}{3}e^{5t} + \frac{13}{3}e^{2t}.$$

□

Example 10.3. *Solve the IVP*

$$2y'' - 5y' + 2y = 0 \qquad y(0) = -3, y'(0) = 3.$$

The characteristic equation is

$$2r^2 - 5r + 2 = 0$$
$$(2r-1)(r-2) = 0.$$

The roots are $r_1 = \frac{1}{2}$ and $r_2 = 2$. Thus the general solution and its derivative are

$$y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{2t}$$
$$y'(t) = \frac{c_1}{2}e^{\frac{1}{2}t} + 2c_2 e^{2t}.$$

Plugging in the initial conditions gives

$$-3 = c_1 + c_2$$
$$3 = \frac{c_1}{2} + 2c_2.$$

This system is solved by $c_1 = -6$ and $c_2 = 3$. So the particular solution is

$$y(t) = -6e^{\frac{1}{2}t} + 3e^{2t}.$$

□

Example 10.4. *Solve the IVP*

$$y'' + 5y' = 0 \qquad y(0) = 2, y'(0) = -5.$$

The characteristic equation is

$$r^2 + 5r = 0$$
$$r(r+5) = 0$$

and this has roots $r_1 = 0$ and $r_2 = -5$. The general solution and its derivative are

$$y(t) = c_1 + c_2 e^{-5t}$$
$$y'(t) = -5c_2 e^{-5t}.$$

Using the initial conditions yields

$$2 = c_1 + c_2$$
$$-5 = -5c_2$$

which has solution $c_1 = 1$ and $c_2 = 1$, So the particular solution is

$$y(t) = 1 + e^{-5t}.$$

□

Example 10.5. *Solve the IVP*

$$y'' - 2y' - 8 = 0 \qquad y(2) = 1, y'(2) = 0.$$

The characteristic equation is

$$r^2 - 2r - 8 = 0$$
$$(r-4)(r+2) = 0$$

so the roots are $r_1 = 4$ and $r_2 = -2$. The general solution and its derivative are

$$y(t) = c_1 e^{4t} + c_2^{-2t}$$
$$y'(t) = 4c_1 e^{4t} - 2c_2 e^{-2t}$$

and plugging in the initial conditions yields

$$1 = c_1 e^8 + c_2 e^{-4}$$
$$0 = 4c_1 e^8 - 2c_2 e^{-4}.$$

The solution is $c_1 = \frac{2}{6}e^{-8}$ and $c_2 = \frac{4}{6}e^4$ and the particular solution is

$$y(t) = \frac{1}{3e^8}e^{4t} + \frac{2e^4}{3}e^{-2t}.$$

□

Example 10.6. *Find the general solution to the DE*

$$y'' + y' - 3y = 0.$$

The characteristic equation is

$$r^2 + r - 3 = 0$$

which has roots

$$r_{1,2} = \frac{-1 \pm \sqrt{13}}{2}.$$

Thus the general solution is

$$y(t) = c_1 e^{\frac{-1+\sqrt{13}}{2}} + c_2 e^{\frac{-1-\sqrt{13}}{2}}.$$

□

2. Complex Roots

Now suppose the characteristic equation 10.2 has complex roots of the form $r_{1,2} = \alpha \pm i\beta$. By our earlier discussion, this means that we have the following two solutions to our original differential equation 10.1:

$$y_1(t) = e^{(\alpha+i\beta)t} \qquad y_2(t) = e^{(\alpha-i\beta)t}.$$

This is problematic, because both $y_1(t)$ and $y_2(t)$ are complex-valued. Since our original equation was both simple and had real coefficients, it would be ideal if we could find two real valued "different" enough solutions so that we can form a real-valued general solution. How do we do this?

Euler's Formula. Euler's Formula says

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

In other words, we can write an imaginary exponential as a sin and a cos. How do we establish this? There are two nice ways of seeing this fact.

Differential Equations. First, we want to write $e^{i\theta} = f(\theta) + ig(\theta)$. We also have

$$f' + ig' = \frac{d}{d\theta}\left[e^{i\theta}\right] = ie^{i\theta} = if - g.$$

Thus $f' = -g$ and $g' = f$, so $f'' = -f$ and $g'' = -g$. Since $e^0 = 1$, we know that $f(0) = 1$ and $g(0) = 0$. We conclude that $f(\theta) = \cos(\theta)$ and $g(\theta) = \sin(\theta)$, so

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Taylor Series. Recall that the Taylor series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

while the Taylor series for $\sin(x)$ and $\cos(x)$ are

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

If we set $x = i\theta$ in the first series, we get

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

So we can write our two complex exponentials as

$$e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)$$
$$e^{(\alpha+i\beta)t} = e^{\alpha t}e^{-i\beta t} = e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right)$$

where the minus sign pops out of the sign in the second equation due to the oddness of sin. Notice that our new expressions are still complex. However, using the Principle of Superposition, we can obtain the following two solutions:

$$y_1(t) = \frac{1}{2}\left(e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\right) + \frac{1}{2}\left(e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))\right) = e^{\alpha t}\cos(\beta t)$$
$$y_2(t) = \frac{1}{2i}\left(e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\right) - \frac{1}{2i}\left(e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))\right) = e^{\alpha t}\sin(\beta t).$$

Exercise. Check that $y_1(t) = e^{\alpha t}\cos(\beta t)$ and $y_2(t) = e^{\alpha t}\sin(\beta t)$ are in fact solutions to the differential equation 10.1 when the roots of 10.2 are $\alpha \pm i\beta$.

So now we have two solutions $y_1(t)$ and $y_2(t)$ which are real-valued. It turns out that they're also "different" enough, so, if the roots to the characteristic equation are $r_{1,2} = \alpha \pm i\beta$, we have the general solution

$$y(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t).$$

Let's do some examples.

EXAMPLE 10.7. *Solve the IVP*

$$y'' - 4y' + 9y = 0 \qquad y(0) = 0, y'(0) = -2.$$

The characteristic equation is

$$r^2 - 4r + 9 = 0$$

which has roots $r_{1,2} = 2 \pm i\sqrt{5}$. Thus the general solution and its derivative are

$$\begin{aligned} y(t) &= c_1 e^{2t}\cos(\sqrt{5}t) + c_2 e^{2t}\sin(\sqrt{5}t) \\ y'(t) &= 2c_1 e^{2t}\cos(\sqrt{5}t) - \sqrt{5}c_1 e^{2t}\sin(\sqrt{5}t) + 2c_2 e^{2t}\sin(\sqrt{5}t) + \sqrt{5}c_2 e^{2t}\cos(\sqrt{5}t). \end{aligned}$$

If we apply the initial conditions, we get

$$\begin{aligned} 0 &= c_1 \\ -2 &= 2c_1 + \sqrt{5}c_2 \end{aligned}$$

which is solved by $c_1 = 0$ and $c_2 = -\frac{2}{\sqrt{5}}$. So the particular solution is

$$y(t) = -\frac{2}{\sqrt{5}}e^{2t}\sin(\sqrt{5}t).$$

□