LECTURE 32

Systems With Real Eigenvalues

1. Solutions to Systems of DEs

Now that we have the required background, we can return our attention to homogeneous systems of first order differential equations (*i.e.*, there shouldn't be any terms involving just the independent variable t). A two-dimensional linear system of differential equations has the form

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned}$$

Suppose we've got our system written in matrix form,

$$\boldsymbol{x}' = A\boldsymbol{x}. \tag{32.1}$$

How do we go about solving this equation? If A were a 1×1 matrix, *i.e.*, a constant, and x were a vector with 1 component, the differential equation would be the separable equation

$$x' = ax.$$

We know that this is solved by

$$x(t) = ce^{at}.$$

We might think, then, that in the $n \times n$ case, instead of a we have some other constant in the exponential, and instead of the constant of integration c we have some constant vector $\boldsymbol{\eta}$.

So let's guess that the form of a solution will be

$$\boldsymbol{x}(t) = \boldsymbol{\eta} e^{rt}. \tag{32.2}$$

Plugging this guess into (32.1) gives

$$\begin{aligned} r\boldsymbol{\eta} e^{rt} &= A\boldsymbol{\eta} e^{rt} \\ (A\boldsymbol{\eta} - r\boldsymbol{\eta})e^{rt} &= \mathbf{0} \\ (A - rI)\boldsymbol{\eta} e^{rt} &= \mathbf{0}. \end{aligned}$$

As $e^{rt} \neq 0$, we end up with the requirement that

$$(A - rI)\boldsymbol{\eta} = \mathbf{0}.$$

This should look familiar; it's the condition for $\boldsymbol{\eta}$ to be an eigenvector of A with eigenvalue r. Thus, we conclude that for (32.2) to be a solution to (32.1), we must have $\boldsymbol{\eta}$ an eigenvalue of A with eigenvalue r.

That tells us how to get some solutions to systems of differential equations: we find the eigenvalues and vectors of the coefficient matrix A, then form solutions using (32.2). But how do we form the general solution?

Thinking back to the second/higher order linear case, what we need are enough linearly independent solutions to form a fundamental set. As we noticed last lecture, if we have all simple eigenvalues, then we're fine: the eigenvectors are all linearly independent, and so the solutions formed will be as well. Things will get more complicated when we have eigenvalues of higher multiplicity, but we can deal with that later.

So, we'll find the two fundamental solutions of the form (32.2), then take their linear combination to get our general solution.

2. The Phase Plane

We're going to rely very heavily on qualitatively understanding what solutions to a linear system of differential equations look like; this will pay off when we move on to discussing systems of nonlinear equations. We know that the trivial solution $\boldsymbol{x} = \mathbf{0}$ is always a solution to our homogeneous system $\boldsymbol{x}' = A\boldsymbol{x}$. $\boldsymbol{x} = \mathbf{0}$ is an example of an *equilibrium solution*, *i.e.*, it satisfies

$$\boldsymbol{x}' = A\boldsymbol{x} = 0$$

and is a constant solution. We'll assume that our coefficient matrix A is nonsingular; this implies that $\boldsymbol{x} = \mathbf{0}$ is the only equilibrium solution.

The question we want to ask is whether other solutions move towards or away from this constant solution as $t \to \pm\infty$, so that we can understand the long term behavior of the system. This is no different than what we did when we classified equilibrium solutions for first order autonomous equations; all we're doing is generalizing those ideas to systems of differential equations.

When we drew our solution spaces then, we did so on a $t - y$ plane. This was suitable at that point, but it would be very difficult for us to draw now: to do something analogous, we would require three dimensions, since we would have to sketch both x_1 and x_2 versus t. Instead, what we will do is "ignore" t and think of our solutions as trajectories on the $x_1 - x_2$ plane. Then our equilibrium solution is the origin. The $x_1 - x_2$ plane is called the *phase plane.* We'll see examples of how to sketch trajectories of solutions on the phase plane as we learn how to solve systems of differential equations. Such a sketch of solutions is called the *phase portrait* of the system.

3. Real, Distinct Eigenvalues

Let's get back to the matrix system (32.1). At this point, we know that if λ_1 and λ_2 are real and distinct eigenvalues of the 2×2 coefficient matrix A associated with eigenvectors $\boldsymbol{\eta}^{(1)}$ and $\boldsymbol{\eta}^{(2)}$, respectively. We know that $\boldsymbol{\eta}^{(1)}$ and $\boldsymbol{\eta}^{(2)}$ are linearly independent, as λ_1 and λ_2 are simple. Thus the solutions obtained from them using (32.2) will also be linearly independent, and in fact will form a fundamental set of solutions. The general solution, then, is

$$\boldsymbol{x}(t) = c_1 e^{\lambda_1 t}\boldsymbol{\eta}^{(1)} + c_2 e^{\lambda_2 t}\boldsymbol{\eta}^{(2)}.$$

So, if we have real and distinct eigenvalues, all that we have to do is find the eigenvectors, form the general solution as above, and use any initial conditions that may exist. Let's do some examples. We'll also see how to sketch the phase portrait of the examples.

EXAMPLE 32.1. *Solve the following initial value problem.*

$$\boldsymbol{x}' = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix} \boldsymbol{x} \qquad \boldsymbol{x}(0) = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda - 2)(\lambda + 3) \end{aligned}$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -3$. Next, we find the eigenvectors.

(1) $\lambda_1 = 2$

$$(A - 2I)\boldsymbol{\eta} = \mathbf{0}$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So we'll want to find solutions to the system

$$\begin{aligned} -4\eta_1 + 2\eta_2 &= 0 \\ 2\eta_1 - \eta_2 &= 0. \end{aligned}$$

Using either equation, we find $\eta_2 = 2\eta_1$, and so any eigenvector has the form

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ 2\eta_1 \end{pmatrix}.$$

Choosing $\eta_1 = 1$, we obtain a first eigenvector

$$\boldsymbol{\eta}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(2) $\lambda_2 = -3$

$$(A + 3I)\boldsymbol{\eta} = \mathbf{0}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So we'll want to find solutions to the system

$$\begin{aligned} \eta_1 + 2\eta_2 &= 0 \\ 2\eta_1 + 4\eta_2 &= 0. \end{aligned}$$

Using either equation, we find $\eta_1 = -2\eta_2$, and so any eigenvector has the form

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -2\eta 2 \\ \eta_2 \end{pmatrix}.$$

Choosing $\eta_2 = 1$, we obtain a second eigenvector

$$\boldsymbol{\eta}^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Thus our general solution is

$$\boldsymbol{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Now, we have our initial condition. Let's use it to solve for c_1 and c_2. The condition says

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \boldsymbol{x}(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

All that's left is to write out this matrix equation as a system of equations and then solve.

$$\begin{aligned} c_1 - 2c_2 &= 5 \\ 2c_1 + c_2 &= 0 \end{aligned} \qquad \Rightarrow \qquad c_1 = 1, c_2 = -2$$

Thus the particular solution is

$$\boldsymbol{x}(t) = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2e^{-3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

$\square$

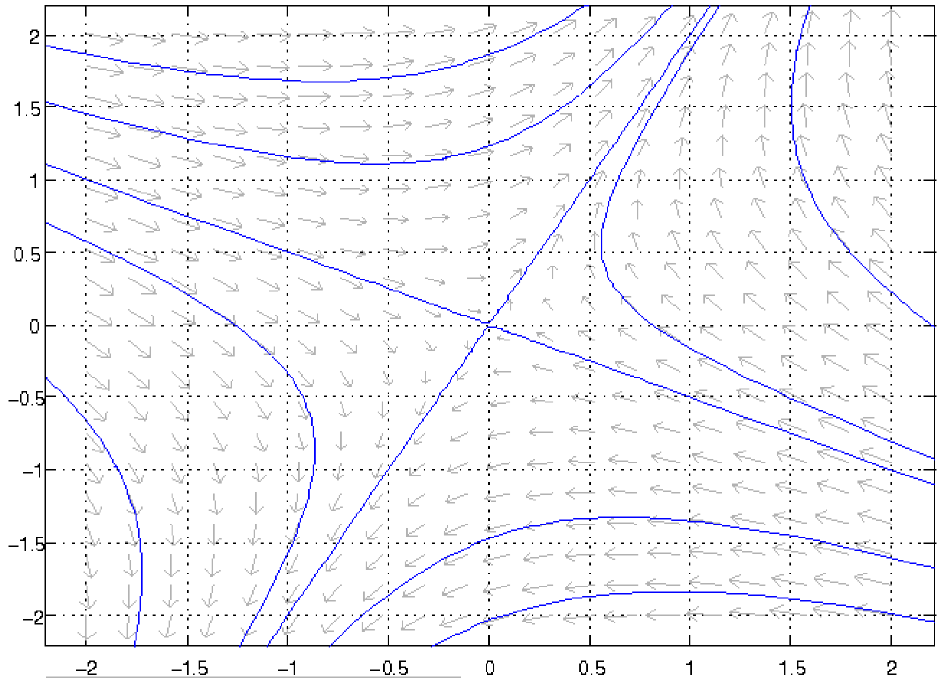

FIGURE 32.1. Phase portrait of the saddle point in Example 32.1.

EXAMPLE 32.2. *Sketch the phase portrait of the system from Example 32.1.*
In the last example, we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$\lambda_1 = 2 \qquad \boldsymbol{\eta}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -3 \qquad \boldsymbol{\eta}^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

The starting point for the phase portrait involves sketching solutions corresponding to the eigenvectors (*i.e.*, with c_1 or $c_2 = 0$). We know that if $\boldsymbol{x}(t)$ is one of these solutions,

$$\boldsymbol{x}'(t) = Ac_i e^{\lambda_i t}\eta^{(i)} = c_i\lambda_i e^{\lambda_i t}\eta^{(i)}.$$

This is just, for any t, a constant times the eigenvector, which indicates that lines in the direction of the eigenvector are these solutions to the system. These are called the *eigensolutions* of the system.

Next, we need to consider the direction that these solutions move in. Let's start with the first eigensolution, which corresponds to the solution with $c_2 = 0$. The first eigenvalue is $\lambda_1 = 2 > 0$. This indicates that this eigensolution will grow exponentially, as the exponential in the solution has a positive exponent. The second eigensolution corresponds to $\lambda_2 = -3 < 0$, so the exponential in the appropriate solution is negative. Hence this solution will decay and move towards the origin.

What does a typical trajectory do (*i.e.*, a trajectory where both $c_1, c_2 \neq 0$)? The general solution is

$$\boldsymbol{x}(t) = c_1 e^{2t}\boldsymbol{\eta}^{(1)} + c_2 e^{-3t}\boldsymbol{\eta}^{(2)}.$$

Thus as $t \to \infty$, this solution will limit to the positive eigensolution, as the component corresponding to the negative eigensolution will decay away. On the other hand, as $t \to -\infty$, the trajectory will asymptotically reach the negative eigensolution, as the positive eigensolution component will be tiny.

The end result is the phase portrait as in Figure 32.1. When the phase portrait looks like this (which happens whenever the eigenvalues have mixed signs), the equilibrium solution at the origin is classified as a *saddle point* and is *unstable*. ☐

Example 32.3. *Solve the following initial value problem.*

$$\begin{aligned} x_1' &= 4x_1 + x_2 \qquad & x_1(0) &= 6 \\ x_2' &= 3x_1 + 2x_2 \qquad & x_2(0) &= 2 \end{aligned}$$

Before we can solve anything, we need to convert this system into matrix form. Doing so converts the initial value problem to

$$\boldsymbol{x}' = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \boldsymbol{x} \qquad \boldsymbol{x}(0) = \begin{pmatrix} 6 \\ 2 \end{pmatrix}.$$

To solve, the first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} \\ &= \lambda^2 - 6\lambda + 5 \\ &= (\lambda - 1)(\lambda - 5) \end{aligned}$$

So the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 5$. Next, we find the eigenvectors.

(1) $\lambda_1 = 1$

$$(A - I)\boldsymbol{\eta} = \mathbf{0}$$
$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So we'll want to find solutions to the system

$$\begin{aligned} 3\eta_1 + \eta_2 &= 0 \\ 3\eta_1 + \eta_2 &= 0. \end{aligned}$$

Using either equation, we find $\eta_2 = -3\eta_1$, and so any eigenvector has the form

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ -3\eta_1 \end{pmatrix}.$$

Choosing $\eta_1 = 1$, we obtain a first eigenvector

$$\boldsymbol{\eta}^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

(2) $\lambda_2 = 5$

$$(A - 5I)\boldsymbol{\eta} = \mathbf{0}$$
$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So we'll want to find solutions to the system

$$\begin{aligned} -\eta_1 + \eta_2 &= 0 \\ 3\eta_1 - 3\eta_2 &= 0. \end{aligned}$$

Using either equation, we find $\eta_1 = \eta_2$, and so any eigenvector has the form

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta 2 \\ \eta_2 \end{pmatrix}.$$

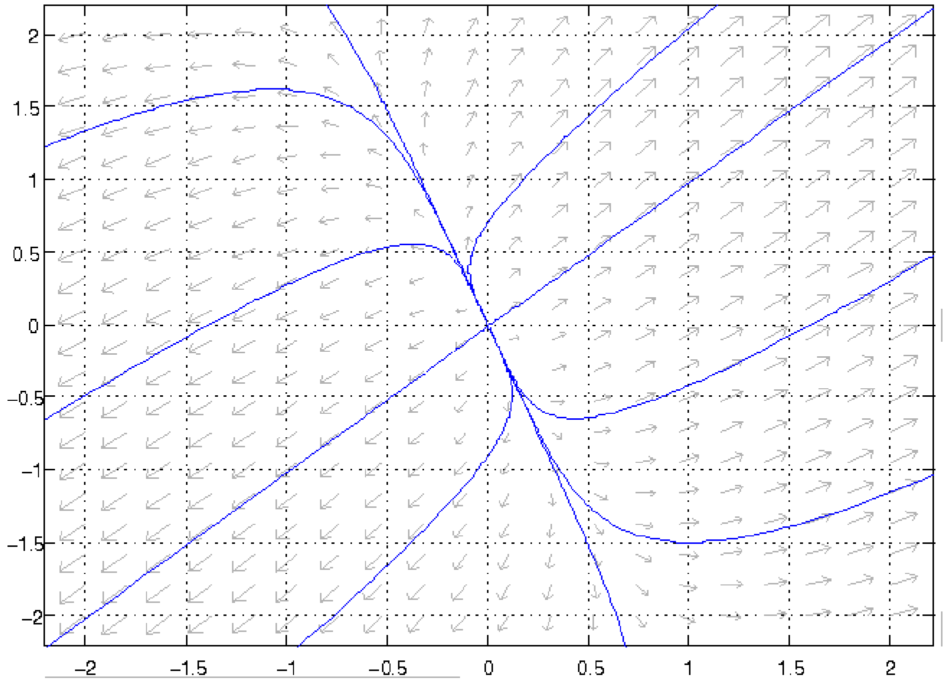

FIGURE 32.2. Phase portrait of the unstable node in Example 32.3.

Choosing $\eta_2 = 1$, we obtain a second eigenvector

$$\boldsymbol{\eta}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus our general solution is

$$\boldsymbol{x}(t) = c_1 e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now, we have our initial condition. Let's use it to solve for c_1 and c_2. The condition says

$$\begin{pmatrix} 6 \\ 2 \end{pmatrix} = \boldsymbol{x}(0) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

All that's left is to write out this matrix equation as a system of equations and then solve.

$$\begin{aligned} c_1 + c_2 &= 6 \\ -3c_1 + c_2 &= 2 \end{aligned} \qquad \Rightarrow \qquad c_1 = 1, c_2 = 5$$

Thus the particular solution is

$$\boldsymbol{x}(t) = e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 5e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$\square$

EXAMPLE 32.4. *Sketch the phase portrait of the system from Example 32.3.*
In the last example, we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$\begin{aligned} \lambda_1 &= 1 & \qquad \boldsymbol{\eta}^{(1)} &= \begin{pmatrix} 1 \\ -3 \end{pmatrix} \\ \lambda_2 &= 5 & \qquad \boldsymbol{\eta}^{(2)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

We begin by sketching the eigensolutions (again, these are the straight lines in the directions of the eigenvectors). Both of these trajectories move away from the origin, though, as the eigenvalues are both positive.

Since $|\lambda_2| > |\lambda_1|$, we call the second eigensolution the *fast eigensolution* and the first one the *slow eigensolution*. The terminology comes from the fact that the eigensolution corresponding to the eigenvalue with larger magnitude will either grow or decay more quickly than the other one.

As both grow in forward time, asymptotically, as $t \to \infty$, the fast eigensolution will dominate the typical trajectory, as it gets larger much more quickly than the slow eigensolution does. So, in forward time, other trajectories will get closer and closer to the eigensolution corresponding to $\boldsymbol{\eta}^{(2)}$. On the other hand, as $t \to -\infty$, the fast eigensolution will decay more quickly than the slow one, and so the eigensolution corresponding to $\boldsymbol{\eta}^{(1)}$ will dominate in backwards time.

Thus the phase portrait will look like Figure 32.2. Whenever we have two positive eigenvalues, every solution moves away from the origin. We call the equilibrium solution at the origin, in this case, a *node* and classify it as being *unstable*. □

Example 32.5. *Solve the following initial value problem.*

$$\begin{aligned} x_1' &= -5x_1 + x_2 & x_1(0) &= 2 \\ x_2' &= 2x_1 - 4x_2 & x_2(0) &= -1 \end{aligned}$$

First, we convert the system intro matrix form.

$$\boldsymbol{x}' = \begin{pmatrix} -5 & 1 \\ 2 & -4 \end{pmatrix} \boldsymbol{x} \qquad \boldsymbol{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

To solve, the first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} -5-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} \\ &= \lambda^2 + 9\lambda + 18 \\ &= (\lambda+3)(\lambda+6) \end{aligned}$$

So the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -6$. Next, we find the eigenvectors.

(1) $\lambda_1 = -3$

$$\begin{aligned} (A + 3I)\boldsymbol{\eta} &= \mathbf{0} \\ \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So we'll want to find solutions to the system

$$\begin{aligned} -2\eta_1 + \eta_2 &= 0 \\ 2\eta_1 - \eta_2 &= 0. \end{aligned}$$

Using either equation, we find $\eta_2 = 2\eta_1$, and so any eigenvector has the form

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ 2\eta_1 \end{pmatrix}.$$

Choosing $\eta_1 = 1$, we obtain a first eigenvector

$$\boldsymbol{\eta}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(2) $\lambda_2 = -6$

$$(A+6I)\boldsymbol{\eta} = \mathbf{0}$$
$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So we'll want to find solutions to the system

$$\begin{aligned} \eta_1 + \eta_2 &= 0 \\ 2\eta_1 + 2\eta_2 &= 0. \end{aligned}$$

Using either equation, we find $\eta_1 = -\eta_2$, and so any eigenvector has the form

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta 2 \\ \eta_2 \end{pmatrix}.$$

Choosing $\eta_2 = 1$, we obtain a second eigenvector

$$\boldsymbol{\eta}^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Thus our general solution is

$$\boldsymbol{x}(t) = c_1 e^{-3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Now, we have our initial condition. Let's use it to solve for c_1 and c_2. The condition says

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \boldsymbol{x}(0) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

All that's left is to write out this matrix equation as a system of equations and then solve.

$$\begin{aligned} c_1 - c_2 &= 2 \\ 2c_1 + c_2 &= -1 \end{aligned} \qquad \Rightarrow \qquad c_1 = \frac{1}{3}, c_2 = -\frac{5}{3}$$

Thus the particular solution is

$$\boldsymbol{x}(t) = \frac{1}{3}e^{-3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{5}{3}e^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

□

Example 32.6. *Sketch the phase portrait of the system from Example 32.5.*

In the last example, we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$\begin{aligned} \lambda_1 &= -3 & \boldsymbol{\eta}^{(1)} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \lambda_2 &= -6 & \boldsymbol{\eta}^{(2)} &= \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \end{aligned}$$

We begin by sketching the eigensolutions. Both of these trajectories decay towards from the origin, though, as the eigenvalues are both negative.

Since $|\lambda_2| > |\lambda_1|$, the second eigensolution is the fast eigensolution and the first one the slow eigensolution.

In the general solution, both exponentials are negative, and so every solution will decay and move towards the origin. Asymptotically, as $t \to \infty$ and the trajectory gets closer and closer to the origin, the slow eigensolution will dominate the typical trajectory, as dies out less quickly than the fast eigensolution. So, in forward time, other trajectories will get closer and closer to the eigensolution corresponding to $\boldsymbol{\eta}^{(1)}$. On the other hand, as $t \to -\infty$, the fast eigensolution will

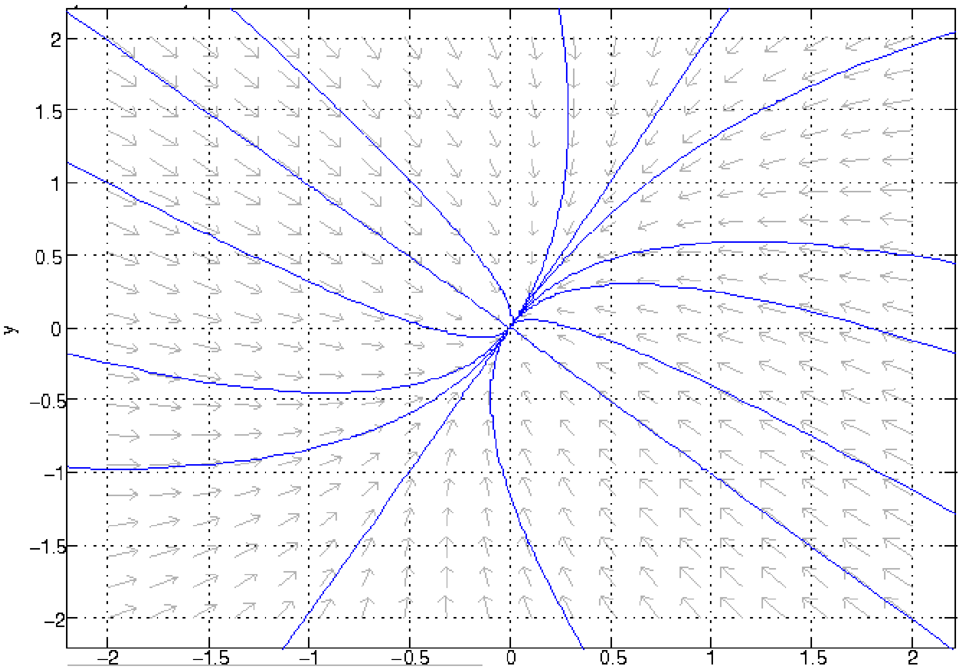

FIGURE 32.3. Phase portrait of the stable node in Example 32.5.

grow more quickly than the slow one, and so the eigensolution corresponding to $\boldsymbol{\eta}^{(2)}$ will dominate in backwards time.

Thus the phase portrait will look like Figure 32.3. Whenever we have two negative eigenvalues, every solution moves towards the origin. We call the equilibrium solution at the origin, in this case, a *node* and classify it as being *asymptotically stable*. $\square$