LECTURE 37

Applications of Nonlinear Systems

In today's lecture, we'll look at some particular nonlinear systems, generally inspired from physics or biology. We'll then use the linearization method we've been discussing for the last few lectures to try to understand the behavior of these systems.

1. Competing Species

One of the classical examples of a nonlinear system is the Lotka-Volterra model of competition between two species (let's take them to be rabbits and kangaroos). Both species are competing for the same food supply (the grass) and there's a limited amount of this resource. We'll assume these populations exist in an environment without predators.

What are the main considerations we'll need to keep in mind to write down this system?

(1) When we discussed the logistic equation as an example of an autonomous first order equation, we mentioned the existence of an environmental carrying capacity: that is, given a species which consumes resources at a certain rate, there is a certain upper limit for the population of this species that the environment can support. Here, since we're assuming that we've got a kangaroo population and a rabbit population and no others, we can assume that in the absence of the other population, each would grow to its carrying capacity. Thus we'll incorporate logistic growth into the equations for each species. We'll also assign rabbits a higher intrinsic growth rate due to the well-known ability of rabbits to reproduce.

(2) What happens when the two species encounter each other while grazing? Sometimes the rabbit might get to eat, but, the kangaroo being significantly larger, they will normally push the rabbit aside and start eating. We can assume that these encounters will occur at a rate proportional to the size of each population. In light of the previous comment, we'll also assume that the conflicts reduce the growth rate for each species, but that the rabbit population is more severely affected.

In particular, if $x(t)$ is the rabbit population at time t and $y(t)$ is the kangaroo population at time t, the model

$$x' = x(3 - x - 2y) \tag{37.1}$$

$$y' = y(2 - x - y) \tag{37.2}$$

incorporates these assumptions.

Our first task is to find the fixed points of this system. We solve $x' = y' = 0$ and obtain four fixed points: $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$. Next, let's use the linearization method to try to classify them. The Jacobian of the system (37.1) is

$$A = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Let's now consider each fixed point to determine the nearby behavior.

- $(0,0)$: Here $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$. This has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$, so we can conclude that $(0,0)$ is an unstable node, as non-degenerate or star nodes are preserved in the original nonlinear system. Recall that right at a node, typical trajectories are tangential to the slow eigensolution, which in this case is the eigensolution corresponding to $\lambda_2 = 2$. The eigenvector corresponding to λ_2 is $\boldsymbol{\eta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so trajectores will leave the node along the y-axis.

- $(0, 2)$: Here $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$. This matrix has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$; hence the fixed point is a stable node. A typical trajectory approaches it along the eigensolution corresponding to λ_1, which is in the direction of the eigenvector $\boldsymbol{\eta} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
- $(3, 0)$: Here $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$. The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$, so this is also a stable node. The slow eigensolution is in the direction of the eigenvector of λ_2, namely $\boldsymbol{\eta} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.
- $(1, 1)$: Here $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$ and $\lambda_{1,2} = -1 \pm \sqrt{2}$. This is a saddle point.

We can further note that the x-axis is a straight-line trajectory, as $x' = 0$ when $x = 0$. Similarly, the y-axis is a straight-line trajectory. Lines where $x' = 0$ or $y' = 0$ are called *nullclines*, and in this case these nullclines are also solutions. Putting this all together and using some common-sense to fill in the rest of the trajectories, we obtain a phase portrait that looks something like Figure 37.1.

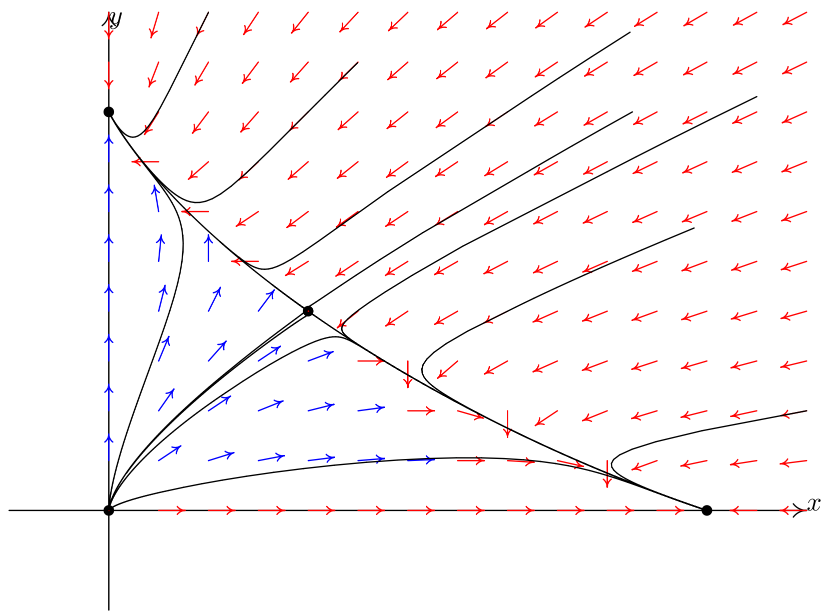

FIGURE 37.1. Phase portrait for the rabbits vs. kangaroo model (37.1). Notice the fixed points and their stability.

This has an interesting biological interpretation: it shows that, in general, one species drives the other to extinction. If we start above the stable solution of the saddle point, the kangaroos drive the rabbits to extinction, but if we start below it, the rabbits drive the kangaroos to extinction. This phenomenon shows up in more complex models of competing species as well, and it has led to the formulation of the *principle of competitive exclusion*, which states that, in general, two species competing for the same resources cannot coexist. This is why releasing pets into the wild is a bad idea; its species might drive native populations competing for the same resources out.

2. Nonlinear Pendulum

In the absence of any damping or external driving, the motion of a pendulum is governed by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0,$$

where $\theta(t)$ is the angle from the downward vertical, g is acceleration due to gravity, and L is the length of the pendulum.

This is derived from the rotational formulation of Newton's Second Law,

$$\tau = I\alpha,$$

where $\tau = -mgL\sin(\theta)$ is the torque, $I = mL^2$ is the moment of inertia, and $\alpha = \frac{d^2\theta}{dt^2}$ is the rotational acceleration. Let's write $\omega_0 = \sqrt{\frac{g}{L}}$. Then our equation is

$$\theta'' + \omega_0^2 \sin(\theta) = 0. \tag{37.3}$$

For very small angles, this equation will be linearized using $\sin(\theta) \approx \theta$ (this is generally done in high school, for example), but using phase plane methods, we can study the nonlinear equation even for large angles.

Writing (37.3) as a system yields

$$\begin{aligned} \theta' &= \nu \\ \nu' &= -\omega_0^2 \sin(\theta), \end{aligned} \tag{37.4}$$

where ν is the angular velocity.

The fixed points are of the form $(n\pi, 0)$, where n is any integer. The Jacobian of the system is

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_0\cos(\theta) & 0 \end{pmatrix}.$$

Notice that there is no difference between angles that differ by 2π, either physically or formally, so we'll focus on the two fixed points $(0, 0)$ and $(\pi, 0)$.

Near $(0, 0)$, the linearized system is

$$\boldsymbol{x} = \begin{pmatrix} 0 & 1 \\ -\omega_0 & 0 \end{pmatrix}\boldsymbol{x}.$$

The eigenvalues of the coefficient matrix are $\lambda_{1,2} = \pm\omega_0 i$, which says that the linearized system has a center. As we discussed last class, however, that doesn't mean we have a nonlinear center at the fixed point; we could have a spiral. So what do we do?

It turns out that the system (37.4) is an example of a *reversible system*; that is, if we "reverse time" by changing t to $-t$ and ν to $-\nu$ (since reversing time will reverse the velocity, as well), the system stays the same. Physically, this should make sense: if we tape a pendulum's motion and play it backwards, we won't see any physical absurdities.

It turns out that for reversible systems, if the origin is a linear center, it will be a nonlinear center, as well. The idea is that we take a trajectory close to the origin which swirls around the origin (as the origin will have to be a spiral or a center). Reversing time and velocity reflects this to a twin trajectory with the same endpoints but with the arrow reversed, which closes the orbit. This is illustrated in Figure 37.2. Thus $(0, 0)$, and hence $(2k\pi, 0)$, are all centers.

What happens near $(\pi, 0)$? The linearized system is

$$\boldsymbol{x} = \begin{pmatrix} 0 & 1 \\ \omega_0 & 0 \end{pmatrix}\boldsymbol{x}.$$

The coefficient matrix has eigenvalues $\lambda_{1,2} = \pm\sqrt{\omega_0}$. Hence it, and any fixed point of the form $((2k+1)\pi, 0)$, is a saddle point.

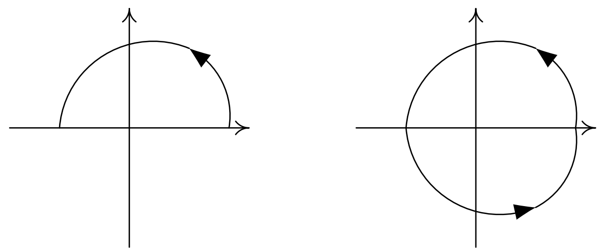

FIGURE 37.2. Why an origin that is a linear center is also a nonlinear center in a reversible system; the trajectory to the left also has its mirror image as a trajectory.

Now we can draw the phase portrait near the fixed points. How do we physically interpret the classification we just found? The centers correspond to a state of neutrally stable equilibrium; the pendulum is at rest and is hanging straight down (as $\theta = 2k\pi$). If we move the pendulum slightly away from there (and possibly give it a little initial velocity), we'll get oscillation back and forth. The saddles, on the other hand, correspond to the cases where the pendulum is at rest but is balanced perfectly up. They're unstable, as if we move the pendulum slightly from this balance, they'll swing back down.

What happens away from the fixed points? This corresponds to giving the pendulum a lot of initial velocity. We'll actually end up with the pendulum rotating around and around the axis, so that the phase portrait looks like Figure 37.3.

EXERCISE (Free Damped Pendulum). If we add a damper with damping coefficient $\gamma > 0$ to the nonlinear pendulum modeled by (37.3), the new equation governing the motion of the pendulum is

$$\theta'' + \gamma\theta' + \omega_0 \sin(\theta) = 0.$$

After writing this equation as a system of first order equations, find and classify the fixed points for all $\gamma > 0$ and plot the phase portraits for each qualitatively different case.

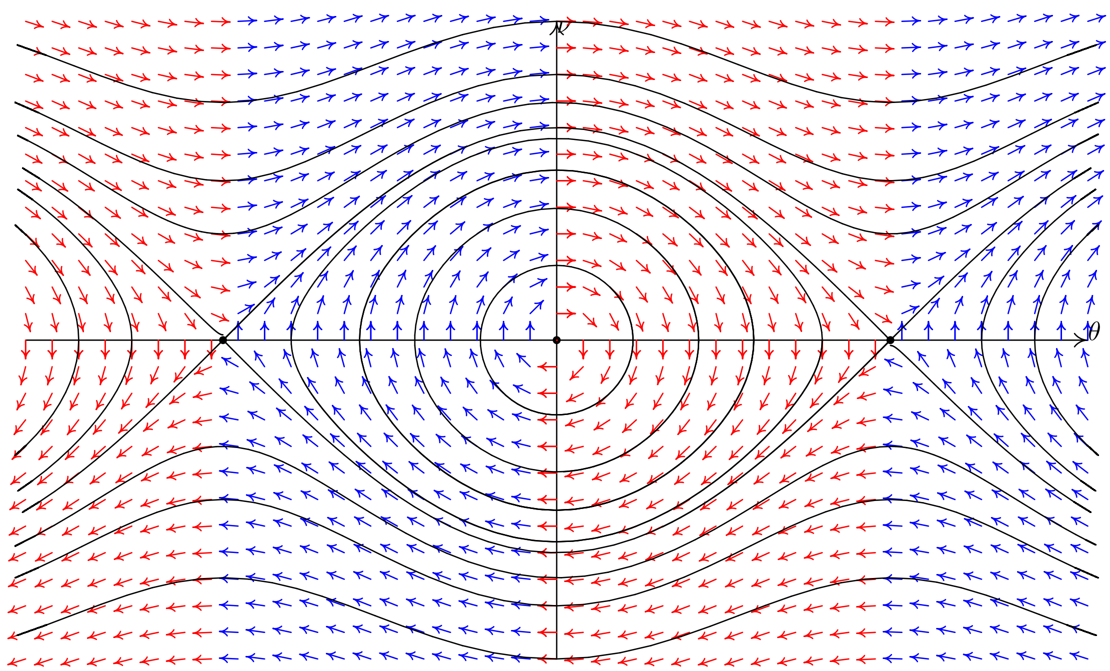

FIGURE 37.3. Phase portrait for the free, undamped nonlinear pendulum (37.4).

3. Next Steps

We've only dipped our toes into the theory of nonlinear systems, but hopefully we've gotten the sense that there's a lot of interesting phenomena occuring here. Let's briefly list some of the directions there are go in from here; most of these would get at least a brief discussion in a second course in differential equations.

3.1. Limit Cycles and Periodic Solutions. After equilibria, the next most important feature of nonlinear systems is the possibility of periodic solutions. These also very strongly affect the overall phase portrait. There are several global methods that can be brought to bear on the problem of detecting and understanding periodic solutions.

There are many questions that can be asked here. If we have a closed trajectory, must it always circle a fixed point? What kinds of fixed points are permitted inside of one? Is there a restriction on the number of closed trajectories we can have or where they can be located?

In fact, we've seen examples where closed trajectories occur in nested families. They can also be isolated: such a solution is called a *limit cycle*. Nearby solutions, which aren't closed, can limit to or away from the limit cycle. Stable limit cycles are very important scientifically, as they model self-sustained oscillations, such as in the human heart. These are, as we've seen, inherently nonlinear phenomena; if a linear system has closed solutions, so are nearby solutions.

3.2. Bifurcations. We've only looked at what happens if we have a single system of nonlinear differential equations. What if we have a family of them, differing by some parameter? For example, what if we have a nonlinear pendulum or an electrical system driven by a constant external force, which we proceed to increase? Can the behavior change as we vary the parameter?

The answer is yes. Many different things can happen here: the number or stability of fixed points or closed orbits can change, for example. Such a change is called a *bifurcation*. This is very important if you think about an externally driven electrical circuit, for example: understanding where and what bifurcations can occur will tell us how much force we can safely apply to the circuit.

Bifurcations can also occur for one-dimensional equations, as well, and some very interesting ones occur for *discrete* systems, where we take some function and iterate it repeatedly, looking for stable and unstable fixed points.

3.3. Chaos. It can be shown that for a two-dimensional system, things behave generally nicely and limit cycles are "typical" in some sense. However, in three dimensions, all bets are off. In 1963, while modeling "convection rolls" in the upper atmosphere, Edward Lorenz wrote down the following system of differential equations:

$$\begin{aligned} x' &= -ax + by \\ y' &= -xz + rx - y \\ z' &= xy - bz, \end{aligned}$$

where a, b are some constants and $r > \frac{a}{b}$. This looks like a relatively simple system, but it turns out to be surprisingly complicated. The solutions don't ever settle down to a periodic orbit or fixed point, but they also don't run off to infinity. Instead, they wrap around two equilibria in a fairly crazy manner.

The manner in which they wrap around these equilibria depends very strongly on the initial conditions; small changes lead to solutions which behave very differently. This is known as *sensitivity to initial conditions*, and it's one of the hallmarks of a chaotic system. It's become known as the "butterfly effect": a butterfly flaps its wings in Beijing, and in New York it rains a few days later instead of being sunny[1]. It's why we can't predict the weather very far in advance, since instruments and computers only have a certain number of decimal places they can measure/compute to, but down the line, the small bits that are lost in the process become very important.

[1]There's another possibility for why this is called the "butterfly effect." There's a classic science fiction story (and an episode of *The Simpsons* parodying it) about a time traveler who goes back in time to hunt dinosaurs and steps on a butterfly. When he returns to his time, everything has changed in a substantial way. Killing that one butterfly changed the "initial conditions" of the universe and upon moving forward a significant amount of time, everything evolved in a different way.