LECTURE 35

# Nonlinear Systems

It's time to turn our attention to systems of equations which actually may show up in applications. These systems, unlike the ones we've been discussing are *nonlinear*; that is, $x_1'$ and $x_2'$ aren't just linear combinations of $x_1$ and $x_2$. As we've seen when looking at single differential equations, nonlinear equations can be difficult to general to deal with. In fact, we won't usually be able to obtain solutions to these systems. Instead, we'll focus more on qualitative analyses of these systems.

We could easily spend an entire semester on this topic. Instead, we'll try to get a bit of the flavor of how these systems differ from the linear systems we've been learning about.

## 1. Introduction to Nonlinear Systems

The general form of a nonlinear two-dimensional system of differential equations is

$$x_1' = f_1(x_1, x_2)$$
$$x_2' = f_2(x_1, x_2)$$

We could rewrite this more compactly in vector notation as

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}),$$

where $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$. For systems like this, there is generally no hope of finding trajectories analytically, as we did for the linear systems we discussed earlier. Thus, as mentioned earlier, our attention will be focused on the qualitative behavior of these solutions.

There are some features of nonlinear phase portraits that are especially salient:

(1) The *fixed* or *critical points*, which are the equilibrium or steady-state solutions. These correspond to points $\boldsymbol{x}$ satisfying $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$; in other words, $x_1$ and $x_2$ are zeroes for both $f_1$ and $f_2$. In this course, we will mostly be focused on learning about the role fixed points play in determining the phase portrait of a nonlinear system.
(2) The *closed orbits*, which correspond to solutions that are periodic for both $x_1$ and $x_2$. We'll briefly discuss some techniques here, but for the most part this is a topic that will be focused on in a future differential equations course you might take.
(3) How trajectories are arranged near fixed points and closed orbits; again, we'll primarily look at what happens near fixed points.
(4) The stability or instability of fixed points and closed orbits; which of these attract nearby trajectories, and which repel them?

How do we even know we have solutions to our general nonlinear system? As we've seen, existence and uniqueness questions can be tricky for nonlinear equations.

Theorem 35.1 (Existence and Uniqueness). *Consider the initial value problem*

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})) \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0.$$

*If $\boldsymbol{f}$ is continuous and so are its partial derivatives $\frac{\partial f_i}{\partial x_i}$ on some region in the plane containing $\boldsymbol{x}_0$, then the initial value problem has a unique solution $\boldsymbol{x}(t)$ on some time interval near $t = 0$.*

The upshot for our purposes? If we have nice enough $f_1$ and $f_2$, so that they and their partial derivatives are continuous for all $x_1$, $x_2$, then any point can be taken as an initial condition for our system.

An important consequence of the existence and uniqueness theorem is that, for the nice systems we'll be considering, *different trajectories can never intersect.* If they did, that point of intersection would be an initial condition corresponding to two different solutions, which can't happen. This means that phase portraits look very polished: they just seem to fit together.

## 2. Linearization Around Critical Points

The starting point for just about any qualitative analysis of nonlinear systems is determining the critical points. These are points $(x_1^*, x_2^*)$ that correspond to equilibrium solutions $x_1(t) = x_1^*$ and $x_2(t) = x_2^*$. As we've discussed, if our system is linear, the only critical point is the origin, $(0, 0)$. Nonlinear systems, however, can have many fixed points, and our goal is to try to determine what we can about the trajectories close to these points.

Consider the system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y). \end{aligned}$$

How do we find these critical points? As they're constant solutions for both $x$ and $y$, they're points where both $x' = 0$ and $y' = 0$. Thus "all" we have to do is to find the values of $x$ and $y$ that are zeroes of both $f$ and $g$. For the examples we'll be looking at, this will be fairly straightforward.

Now suppose $(x_0, y_0)$ is a fixed point. Thus we know that

$$f(x_0, y_0) = g(x_0, y_0) = 0.$$

The goal of the linearization technique is to use our knowledge of linear systems to try to conclude what we can about the phase portrait near $(x_0, y_0)$. To do this, we'll try to approximate our nonlinear system by a linear system, which we can then classify as we've been discussing. Since $(x_0, y_0)$ is a fixed point, and the only fixed point of a linear system is the origin, we'll want to change variables so that $(x_0, y_0)$ becomes the origin of the new coordinate system. Thus, let

$$\begin{aligned} u &= x - x_0 \\ v &= y - y_0. \end{aligned}$$

We need to rewrite our differential equations in terms of $u$ and $v$.

$$\begin{aligned} u' &= x' \\ &= f(x, y) \\ &= f(x_0 + u, y_0 + v) \end{aligned}$$

The natural thing to do here is to Taylor expand $f$ near $(x_0, y_0)$.

$$\begin{aligned} &= f(x_0, y_0) + u\frac{\partial f}{\partial x}(x_0, y_0) + v\frac{\partial f}{\partial y}(x_0, y_0) + \text{higher order terms} \\ &= u\frac{\partial f}{\partial x}(x_0, y_0) + v\frac{\partial f}{\partial y}(x_0, y_0) + \text{H.O.T.} \end{aligned}$$

To simplify notation, we'll sometimes suppress writing explicitly that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are evaluated at the point $(x_0, y_0)$, but it's important to keep this in mind. For our purposes, these partial derivatives are *numbers*, not *functions*.

Another important observation is that, as we're considering what happens very close to our fixed point, $u$ and $v$ are both small; hence the higher order terms are smaller still and will be disgarded in our computations.

Now, by a similar computation we have

$$v' = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + \text{H.O.T.}$$

Ignoring these very small higher order terms, we can write this system of rewritten differential equations in matrix form. The *linearized system* near $(x_0, y_0)$ is

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We will use, from this point on, the notation $f_x = \frac{\partial f}{\partial x}$. The matrix

$$A = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}$$

is called the *Jacobian matrix* at $(x_0, y_0)$ of the vector-valued function $\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f(x_1, x_2) \\ g(x_1, x_2) \end{pmatrix}$. In multivariable calculus, the Jacobian matrix is the appropriate analogue of the single-variable derivative.

We can then study this linear system using the standard techniques.