LECTURE 14

More On The Wronskian

Last lecture, we introduced the Wronskian of two functions y_1 and y_2,

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2'(t)y_1(t).$$

We saw that if $W(y_1, y_2)(t) \neq 0$, then y_1 and y_2 are linearly independant, *i.e.*, the only constants c_1 and c_2 that satisfy

$$c_1 y_1(t) + c_2 y_2(t) = 0$$

are $c_1 = c_2 = 0$. In other words, two functions are linearly independent if they aren't constant multiples of each other.

We also saw that, in the context where y_1 and y_2 are solutions to the linear homogeneous equation

$$p(t)y'' + q(t)y' + r(t)y = 0,$$

$W(y_1, y_2)(t) \neq 0$ is precisely the condition for the general solution of the differential equation to be

$$y(t) = c_1 y_1(t) + c_2 y_2(t),$$

i.e., for y_1 and y_2 to be a fundamental set of solutions. Thus, if y_1 and y_2 are a fundamental set of solutions, they are linearly independent. If we have two solutions y_1 and y_2 which are linearly dependent, on the other hand, then they cannot possibly be a fundamental set of solutions, as they have a zero Wronskian.

There was one point last lecture that we should clear up now. We know that if our initial data is at t_0, y_1 and y_2 will be a fundamental set of conditions if and only if $W(y_1, y_2)(t_0) \neq 0$. But this is a condition only at one point. What happens if y_1 and y_2 have a nonzero Wronskian only at t_0 but not at nearby points? This would be problematic, since then our condition would tell us that y_1 and y_2 are a fundamental set of solutions for initial dat at t_0, but not for any initial data near t_0. The possibility of this should seem odd, since we know there should be a unique solution on any interval around t_0 where we have continuity. So how do we know that this can't happen? The answer is something called Abel's Theorem.

1. Abel's Theorem

You may notice that throughout our entire discussion of the Wronskian, we have yet to actually use the differential equation (beyond deriving the formula for the Wronskian assuming that y_1 and y_2 satisfied some differential equation). Fortunately, when y_1 and y_2 are solutions to a linear homogeneous differential equation, we can say a bit more about their Wronskian.

THEOREM 14.1 (Abel's Theorem). *Suppose $y_1(t)$ and $y_2(t)$ solve the linear homogeneous equation*

$$y''(t) + p(t)y' + q(t)y = 0,$$

where $p(t)$ and $q(t)$ are continuous on some interval (a, b). Then, for $a < t < b$, their Wronskian is given by

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)e^{-\int_{t_0}^{t} p(x)\,dx},$$

where t_0 is in (a, b).

If $W(y_1, y_2)(t_0) \neq 0$ at some point t_0 in the interval (a, b), then Abel's Theorem tells us that the Wronskian can't be zero for any t in (a, b), since exponentials are never zero. This assures us that we can change our initial data (without crossing points of discontinuity of the coefficient functions) without worry that our general solution will change.

Another advantage of Abel's Theorem is that it lets us compute the general form of the Wronskian of any two solutions to the differential equation without knowing them explicitly. This is useful, for example, with regard to reduction of order, where we only begin by knowing a single solution. The formulation given in the statement of the theorem isn't so computationally useful, however, because we might not have a precise t_0 in mind, let alone knowing the value of the Wronskian there. But if we apply the Fundamental Theorem of Calculus, things simplify nicely.

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)e^{-\int_{t_0}^{t} p(x)\, dx} = ce^{-\int p(t)\, dt}$$

What is this constant c? Well, it doesn't really end up mattering. If we know the value of the Wronskian at one point, we can compute it, but our general interest in the Wronskian mostly involves knowing its general form. As long as we know $c \neq 0$, that's all that matters to us.

Example 14.1. *Compute, up to a constant, the Wronskian of two solutions y_1 and y_2 of the differential equation*

$$t^4 y'' - 2t^3 y' - t^8 y = 0.$$

First, we need to put the equation in the form specified in Abel's Theorem. We do this by dividing by the leading coefficient.

$$y'' - \frac{2}{t} y' - t^4 y = 0.$$

So, Abel's Theorem tells us

$$W = ce^{-\int -\frac{2}{t}\, dt} = ce^{2 \ln t} = ct^2.$$

□

Ok, great...but the main virtue of this is that it gives us a second way to compute the Wronskian. A general rule in mathematics is that whenever you can compute something in two different ways, something good will happen. In this case, we know by Abel's Theorem that

$$W(y_1, y_2)(t) = ce^{-\int p(t)\, dt}.$$

On the other hand, by definition,

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t).$$

Setting these equal, if we know one solution $y_1(t)$, we're left with a first order differential equation for y_2 that we can then solve.

Let's see this with an example of reduction of order we did the traditional way.

Example 14.2. Suppose we want to find the general solution to $2t^2 y'' + ty' - 3y = 0$ and we're given that $y_1(t) = t^{-1}$ is a solution. We need to find a second solution that will form a fundamental set of solutions with y_1. Let's compute the Wronskian both ways.

$$ce^{-\int \frac{1}{2t}\, dt} = W(t^{-1}, y_2)(t) = y_2' t^{-1} + y_2 t^{-2}$$
$$y_2' t^{-1} + y_2 t^{-2} = ce^{-\frac{1}{2}\ln(t)} = ct^{-\frac{1}{2}}$$

This is a first order linear equation with integrating factor $\mu(t) = e^{\int t^{-1}\,dt} = e^{\ln(t)} = t$. Thus

$$\begin{aligned} [ty_2]' &= ct^{\frac{3}{2}} \\ ty_2 &= \frac{2}{5}ct^{\frac{5}{2}} + k \\ y_2(t) &= \frac{2}{5}ct^{\frac{3}{2}} + kt^{-1} \end{aligned}$$

Now, we can choose constants c and k. Notice that k is the coefficient of t^{-1}, which is just $y_1(t)$. So we don't have to worry about that term, and we can take $k = 0$. We can similarly take $c = \frac{5}{2}$, and so we'll get $y_2(t) = t^{\frac{3}{2}}$, which is precisely what we had gotten when we did reduction of order the traditional way. ☐