LECTURE 5

More First Order Modeling

1. Falling Bodies

Let's now consider some object falling to the ground. This body will obey Newton's Second Law of Motion, which we've seen we can write as a first order differential equation for velocity.

$$m\frac{dv}{dt} = F(t, v)$$

where m is the object's mass and F is the net force acting on the body. We'll deal with a simplified situation where the only forces in play are gravity and air resistance.

It's important here to be careful with signs. We'll go through the convention we'll be using and we have to stick to it. The convention that we're going to work with throughout the course is that *downward displacements and forces are positive.* Hence the force due to gravity is given by $F_G = mg$, where $g \approx 10\text{m/s}^2$ is the gravitational constant.

Air resistance, on the other hand, acts against velocity. If the object is moving up, the air resistance force will act downwards, and vice versa. In general, we'll assume that air resistance is linearly dependant on velocity (*i.e.*, $F_A = -\alpha v$, where F_A is the force due to air resistance). This isn't really realistic, but it will make the problem a bit simpler, as you'll see. So we end up with $F(t, v) = F_G + F_A = 10m - \alpha v$, and our differential equation for velocity is

$$m\frac{dv}{dt} = 10m - \alpha v.$$

EXAMPLE 5.1. *A 50 kg object is shot from a cannon straight up with an initial velocity of 10 m/s off the very tip of a bridge. If the air resistance is given by* $5v$*, determine the velocity of the mass at any time* t *and compute the rock's terminal velocity.*

Strictly speaking, there are two phases to this problem: the one where the object is moving upwards, and the one where it's moving downwards. If we look at the forces, though, it turns out we get the same differential equation

$$50v' = 500 - 5v.$$

. The initial condition is $v(0) = -10$, since we shot the object upwards. Our differential equation is linear; in the standard form it's written as

$$v' + \frac{1}{10}v = 10.$$

So we'll use integrating factors. Our integrating factor is $\mu(t) = e^{\frac{t}{10}}$.

$$\begin{aligned}
\frac{d}{dt}\left[e^{\frac{t}{10}}v(t)\right] &= 10e^{\frac{t}{10}} \\
e^{\frac{t}{10}}v(t) &= \int 10e^{\frac{t}{10}}\,dt \\
&= 100e^{\frac{t}{10}} + c \\
v(t) &= 100 + \frac{c}{e^{\frac{t}{10}}} \\
v(0) = -10 &= 100 + c \Rightarrow c = -110
\end{aligned}$$

So the velocity at any time t is given by

$$v(t) = 100 - \frac{110}{e^{\frac{t}{10}}}.$$

What is the terminal velocity of the rock? The terminal velocity is given by the limit of the velocity as $t \to \infty$, which is 100.

Notice that we also could have computed the velocity of the rock when it hit the ground, or known when the rock hit the ground, if we knew the height of the bridge, since then we could have integrated velocity to get position. □

EXAMPLE 5.2. *A 60 kg skydiver jumps out of a plane with no initial velocity. Assuming the magnitude of air resistance is given by* $0.8|v|$, *what is the appropriate initial value problem modeling his velocity?*

Air resistance is an upward force here (since the velocity is moving downwards), while gravity is acting towards. So our force function should be

$$F(t, v) = mg - .8v.$$

Thus our initial value problem is

$$60v' = 60g - .8v \qquad v(0) = 0.$$

□