LECTURE 4

Modeling With First Order Equations

Before we resume our discussion of how to solve certain types of first order equations, let's move to some applications of linear and separable equations. Our goal is to get a rough introduction to the process of modeling: how do we write a differential equation to model (fairly basic) situations and what can the solution tell us?

We've already seen some examples of first order models. In Lecture 1, we saw two examples of differential equations meant to describe physical situations.

Example 4.1 (Radioactive Decay).

$$\frac{dN}{dt} = -\lambda N(t),$$

where $N(t)$ is the number of atoms of a radioactive isotope and $\lambda > 0$ is the decay constant. This equation is separable, and it's easy to see that, if the inital data is $N(0) = N_0$, the solution is

$$N(t) = N_0 e^{-\lambda t}.$$

So we can see that radioactive decay is exponential. □

Example 4.2 (Newton's Law of Cooling). If we immerse a body in an environment with constant temperature E, then if $B(t)$ is the temperature of the body we have

$$\frac{dB}{dt} = \kappa(E - B),$$

where $\kappa > 0$ is a constant related to the material of the body and how it conducts heat. This equation is also separable; we solved it with initial condition $B(0) = B_0$ to get

$$B(t) = E - \frac{E - B_0}{e^{\kappa t}}.$$

□

How do we write down a model given a description of the situation? There are a few different approaches that we'll see in this class:

(1) Remember that we can think of the derivative of a function as its *rate of change.* It's possible that the description of the problem tells us directly what the rate of change is. Newton's Law of Cooling was an example of this; the original quote told us that the rate of change of the body's temperature was proprotional to the difference in temperature between the body and the environment. All we had to do was set the relevant terms equal.

(2) There are also going to be cases where we're not explicitly given the formula for rate of change in the problem, but if we can glean how a function should change from its physical description, we can set the derivative equal to that. The basic premise that we'll be using here is derivative = increase - decrease. Next, we'll see some more in-depth examples of this process. It should be noted that this can only apply to first order equations, since higher order equations can't just be interepreted as "the rate of change is equal to something."

(3) We may just be adapting a known differential equation to a particular situation. Newton's Second Law $F = ma$ is a prime example of this. It's either a first or second order differential equation (for velocity or position, respectively). If we compile the list of forces acting on the object in question, we can just plug them in for F to yield the differential equation for a particular situation. We'll see this approach later this lecture with regard to falling bodies, but it will also appear later in the course concerning harmonic motion and pendulums.

(4) The last possibility is that we are able to determine (via physical principles) two different expressions for some quantity, one or both of which involve derivatives; setting them equal yields the desired differential equation. We'll see this when we discuss partial differential equations later.

It's important that, when faced with a problem, we begin by determining which of these approaches (or other ones that may be out there) is most applicable before proceeding. This is a skill that will prove valuable in the future.

One other very important point: in general, your differential equation should not depend on the initial condition. The initial data will tell you the starting point, but the way the system evolves (which is given by the differential equation) ought not to depend on that. Now, let's look at some particular examples.

1. Interest

These are among the most straightforward problems to write down. Suppose we've got a bank account (or a mortgage, or some other loan, or so on...the particulars don't change much about the equation) that gives $r\%$ interest per year. If I withdraw a constant w dollars per month, what is the differential equation modeling my account's balance?

Let's take our time unit for t to be years, and denote the balance after t years by $B(t)$. $B'(t)$ is the rate of change of my account balance from year to year, so it will be the difference between the amount that's added to my account and the amount that's withdrawn. The only income I'm getting is interest and we know how much I withdraw each year (it's the monthly withdrawal times 12). Hence

$$B'(t) = \frac{r}{100}B(t) - 12w.$$

This is a linear equation, so we could solve it using integrating factors to know the account balance at any time t. Make sure you pay attention to units: we're not withdrawing w dollars per year, we're withdrawing $12w$.

The same setup will work perfectly for something like the following problem.

Example 4.3. *Bill wants to take out a 25 year loan to buy a house. He knows that he can afford, maximum, monthly payments of \$400. If the going interest rate on housing loans is 4%, what is the largest loan Bill can take out that he will be able to pay off in time?*

Let's say the amount that Bill owes at time t (it's convenient here to measure t in years, so we'll do that) is $B(t)$. We want $B(25) = 0$. We also know that the balance will gain 4% interest (*i.e.*, the amount being added to the balance is $0.4B$) and he can make payments totalling \$4800 each year. So the relevant initial value problem should be

$$B'(t) = .04B(t) - 4800 \qquad B(25) = 0.$$

This is a linear equation with standard form

$$B'(t) - .04B(t) = -4800,$$

so the integrating factor is $\mu(t) = e^{\int -.04\,dt} = e^{-0.4t}$.

$$\begin{aligned}
\left(e^{-\frac{4}{100}t}B(t)\right)' &= -4800e^{-\frac{4}{100}t} \\
e^{-\frac{4}{100}t}B(t) &= -4800\int e^{-\frac{4}{100}t}\,dt \\
&= 120000e^{-\frac{4}{100}t} + c \\
B(t) &= 120000 + ce^{\frac{4}{100}t} \\
B(25) = 0 &= 120000 + ce^{\frac{100}{100}} \Rightarrow c = -120000e^{-1} \\
B(t) &= 120000 - 120000e^{\frac{4}{100}t-1}.
\end{aligned}$$

We want to know the size of the loan, which is the amount Bill owes to begin with, hence $B(0)$.

$$B(0) = 120000 - 120000e^{-1} = 120000\left(1 - e^{-1}\right)$$

□

2. Mixing Problems

Suppose we have an efficient mixing tank that can instantly completely mix whatever is inside of it. The tank has some liquid inside of it. Contaminant is being added to the tank at some constant rate and the mixed solution is drained out at a (possibly different) rate. We can ask ourselves what the amount of contaminant is in the tank at any given time.

How do we write down a differential equation to model this process (note that the well-mixing assumption is *very* important for this simplified situation...it's what allows us to use first order ordinary differential equations)? Let's let $P(t)$ be the amount of pollutant (note: *amount* of pollutant, not the concentration; this is critical) in the tank at time t. What information are we given? We know (or are given enough information to compute) the amount of pollutant that is entering and leaving the tank each unit of time. So we can use the second approach from above:

$$\text{Rate of Change of } P(t) \; = \text{Rate of entry of contaminant } \; - \text{Rate of exit of contaminant.}$$

The rate of entry can be determined in a few different ways. We might just be directly adding contaminant (think about using a pipette to add food coloring to a tank of water) or we might be adding solution with a known concentration of contaminant to the tank (in which case the amount would be the concentration times the volume). The problem will make this clear and then we can compute how much contaminant is being added to the tank each time unit (as you'll see shortly). The important thing is that we determine the amount of stuff being added.

What's the rate of exit? Suppose that we're draining the tank at a rate of r_{out}. The amount of contaminant leaving the tank will be the amount contained in the drained solution; this is given by rate × concentration. We know the rate, so we need to compute the concentration. This will just be the concentration of the solution in the tank (hence the importance of having a uniformly mixed solution), which is in turn given by the amount of contaminant in the tank divided by the volume. So the expression we get is

$$\text{Rate of exit of contaminant } \; = \text{Rate of drained solution} \times \frac{\text{Amount of contaminant}}{\text{Volume of tank}}$$

or

$$\text{Rate of exit of contaminant } \; = r_{\text{out}}\frac{P(t)}{V(t)}.$$

What is $V(t)$? To figure that out, we need to know how the volume is changing. Certainly, the volume is decreasing by r_0 each t. Is anything being added to the volume? That depends; if we're adding some solution to the tank at a certain rate r_{in}, that will add to the in-tank volume.

If we're directly adding contaminant not in solution, it won't. So we'll need to make to make this determination when reading the problem, and in the first case, if our initial volume is given by V_0, we'll get $V(t) = V_0 + t(r_{\text{in}} - r_{\text{out}})$, and in the second, $V(t) = V_0 - t\, r_{\text{out}}$.

At the moment, this may seem complicated; it's actually quite straightforward once we've got a problem in hand. So let's consider some examples.

EXAMPLE 4.4. *Suppose a 120 gallon well-mixed tank initially contains 90 lb. of salt mixed with 90 gal. of water. Salt water (with a concentration of 2 lb/gal) comes into the tank at a rate of 4 gal/min. The solution flows out of the tank at a rate of 3 gal/min. How much salt is in the tank when it is full?*

We can immediately write down the expression for the volume $V(t)$. How much liquid is entering the tank each minute? 4 gallons. How much is leaving during the same minute? 3 gallons. So each minute, the volume increases by 1, and we have $V(t) = 90 + (4-3)t = 90 + t$. This tell us that the tank will be full at $t = 30$.

We let $P(t)$ be the amount of salt (in pounds) in the tank at time t. Ultimately, we want t to determine $P(30)$, since this is when the tank will be full. To write down a differential equation, we need to determine the rates at which salt is leaving and entering the tank each minute. How much salt is entering the tank? We have 4 gallons of salt water entering the tank each minute, and each of those gallons has 2 lb. of salt dissolved in it. Hence we're adding 8 lbs. of salt to the tank each minute. How much is exiting the tank? We're draining the solution at a rate of 3 gallons each minute, and we've seen that the concentration of each of those gallons is $P(t)/V(t)$.

$$\begin{aligned}\frac{dP}{dt} &= (4\text{gal/min})\,(2\text{lb/gal}) - (3\text{gal/min})\left(\frac{P(t)\text{lb}}{V(t)\text{gal}}\right)\\ &= 8 - \frac{3P(t)}{90+t}.\end{aligned}$$

This is the differential equation for the amount of salt in the tank; what we need now is an initial condition. That's easy: $P(0) = 90$, as given in the problem. Now we've got our initial value problem; let's solve. Our equation is linear (*i.e.*of the course $y' + p(t)y = q(t)$), with

$$\begin{aligned}p(t) &= \frac{3}{90+t}\\ q(t) &= 8.\end{aligned}$$

Thus we'll want to use the method of integrating factors.

$$\mu(t) = e^{\int \frac{3}{90+t}\,dt} = e^{3\ln(90+t)} = (90+t)^3$$

$$\begin{aligned}\frac{d}{dt}\left[(90+t)^3 P(t)\right] &= 8(90+t)^3\\ (90+t)^3 P(t) &= \int 8(90+t)^3\,dt\\ &= 2(90+t)^4 + c\\ P(t) &= 2(90+t) + \frac{c}{(90+t)^3}\\ P(0) = 90 &= 2(90) + \frac{c}{90^3} \Rightarrow c = -(90)^4\end{aligned}$$

So we end up with

$$P(t) = 2(90+t) - \frac{90^4}{(90+t)^3}.$$

What did we originally want to know? We wanted to know $P(30)$, which is the amount of salt in the tank when the tank is full. So we can evaluate that:

$$P(30) = 240 - \frac{90^4}{120^3} = 240 - 90\left(\frac{3}{4}\right)^3 = 240 - 90\left(\frac{27}{64}\right).$$

Notice that we could've asked for the concentration at the overflow time, or for that matter, how much salt was in the tank at any particular time before overflow. Our final computation would have been the only difference. □

Exercise. What is the concentration of the tank in Example 4.4 when the tank is full?

For future examples, I won't be quite so verbose when explaining the various steps. The process is basically the same for every mixing problem you're going to encounter; the only differences are the tedium involved in solving the equation and what exactly ought to be computed at the final step.

Example 4.5. *A full 20 liter tank has 20 grams of yellow food coloring dissolved in it. If a yellow food coloring solution (with a concentration of 2 grams/liter) is piped into the tank at a rate of 3 liters/minute while the well mixed solution is drained out of the tank at a rate of 3 liters/minute, what is the limiting concentration of yellow food coloring solution in the tank?*

Our differential equation in this case will be

$$\begin{aligned}\frac{dP}{dt} &= (3\text{l/min})\,(2\text{g/l}) - (3\text{l/min})\,\frac{P(t)\text{g}}{V(t)\text{l}} \\ &= 6 - \frac{3P}{20}\end{aligned}$$

Notice that the volume is constant, since we are draining the tank at the same rate that we're adding solution.

The integrating factor in this case is

$$\mu(t) = e^{\int \frac{3}{20}\,dt} = e^{\frac{3}{20}t}$$

and our initial condition is $P(0) = 20$, so we solve:

$$\begin{aligned}\frac{d}{dt}\left[e^{\frac{3}{20}t}P(t)\right] &= 6e^{\frac{3}{20}t} \\ e^{\frac{3}{20}t}P(t) &= \int 6e^{\frac{3}{20}t}\,dt \\ &= 40e^{\frac{3}{20}t} + c \\ P(t) &= 40 + \frac{c}{e^{\frac{3}{20}t}} \\ P(0) = 20 &= 40 + c \Rightarrow c = -20\end{aligned}$$

and we conclude

$$P(t) = 40 - \frac{20}{e^{\frac{3}{20}t}}.$$

Now, we want to know what is going to happen to the concentration in the limit, or as $t \to \infty$. We know the volume will always be 20 liters.

$$\lim_{t\to\infty} \frac{P(t)}{V(t)} = \lim_{t\to\infty} \frac{40 - 20e^{-\frac{3}{20}t}}{20} = 2.$$

So the limiting concentration is 2 g/l. Why does this make physical sense? □

It turns out that the same process will work (but be slightly more complicated to solve) if the concentration of the incoming solution is variable.

EXAMPLE 4.6. *A 150 gallon tank has 60 gallons of water with 5 pounds of salt dissolved in it. Water with a concentration of* $2+\cos(t)$ *lbs/gal comes into the tank at a rate of 9 gal/hour. If the well mixed solution leaves the tank at a rate of 6 gal/hour, how much salt is in the tank when it overflows?*

The only difference between this and our previous examples is that the incoming concentration is variable. We now write down our differential equation, given that the volume starts at 600 gal and increases at a rate of 3 gal/min.

$$\frac{dP}{dt} = 9\left(2+\cos(t)\right) - \frac{6P}{60+3t}$$

Our initial condition is $P(0)=5$ and our integrating factor will be

$$\mu(t) = e^{\int \frac{6}{60+3t}\,dt} = e^{2\ln(20+t)} = (20+t)^2 .$$

$$\begin{aligned}
\frac{d}{dt}\left[(20+t)^2 P(t)\right] &= 9\left(2+\cos(t)\right)(20+t)^2 \\
(20+t)^2 P(t) &= \int 9\left(2+\cos(t)\right)(20+t)^2\,dt \\
&= 9\left(\frac{2}{3}(20+t)^3 + (20+t)^2\sin(t) + 2(20+t)\cos(t) - 2\sin(t)\right) + c \\
P(t) &= 9\left(\frac{2}{3}(20+t) + \sin(t) + \frac{2\cos(t)}{20+t} - \frac{2\sin(t)}{(20+t)^2}\right) + \frac{c}{(20+t)^2}
\end{aligned}$$

Now we apply the initial condition to get the constant c.

$$\begin{aligned}
P(0) = 5 &= 9\left(\frac{2}{3}(20) + \frac{2}{20}\right) + \frac{c}{400} \\
&= 120 + \frac{9}{10} + \frac{c}{400} \\
c &= -46360.
\end{aligned}$$

We want to know how much salt is in the tank when it overflows. This happens when the volume hits 150, or at $t=30$.

$$\begin{aligned}
P(30) &= 300 + 9\sin(30) + \frac{18\cos(30)}{50} - \frac{18\sin(30)}{2500} - \frac{46360}{2500} \\
&\approx 272.63 \text{ pounds}
\end{aligned}$$

□

We could make a more complicated problem by assuming, *e.g.*, that there will be a change in the situation if the solution ever reached a critical concentration. The process would still be the same; we would just need to solve two different but linked initial value problems.