LECTURE 36

Linearization Near Critical Points

Last lecture, we began considering the nonlinear system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y). \end{aligned} \tag{36.1}$$

We saw that near the critical points (points (x_0, y_0) satisfying $f(x_0, y_0) = g(x_0, y_0) = 0$), which correspond to equilibrium solutions of the system, we can approximate the nonlinear system (36.1) by the linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{36.2}$$

We can then analyze these linear systems to conclude what we can about the behavior of solutions near the equilibria. Notice that this is entirely a local method: we don't obtain information about what happens far away from the critical points. However, we can sometimes use common sense to determine other features of the phase portrait.

EXAMPLE 36.1. *Find the critical points of the following systems, then determine the linearizations near them.*

(i) $x' = y^2 - 3x + 2 \qquad y' = x^2 - y^2$

The first task is to find the critical points. We do this by setting $x' = f(x,y)$ and $y' = g(x,y)$ and then looking for the values of x and y that make both f and g zero.

Looking at the two equations, we might decide that it's easy to conclude something from $g(x,y) = x^2 - y^2 = 0$. Indeed, we see that we must have

$$x^2 = y^2.$$

Thus, the equation $f(x,y) = y^2 - 3x + 2 = 0$ becomes

$$\begin{aligned} x^2 - 3x + 2 &= 0 \\ (x-2)(x-1) &= 0. \end{aligned}$$

So we must have $x = 2$ or $x = 1$. Now, we know $x^2 = y^2$. So if $x = 2$, $y^2 = 4$, and so $y = \pm 2$. Similarly, if $x = 1$, $y^2 = 1$, and so we get $y = \pm 1$.

Summarizing, we have four critical points: $(1,1), (1,-1), (2,2), (2,-2)$.

Now, let's find the linearizations of the system near each of these points. The Jacobian matrix of the nonlinear system at a general point (x,y) is

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -3 & 2y \\ 2x & -2y \end{pmatrix}.$$

To get the coefficient matrix of our linearized system near one of our critical points, all we have to do is evaluate these terms at the point. Thus we get the following linearizations.

$$(1,1): \qquad \boldsymbol{x}' = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} \boldsymbol{x}$$

$$(-1,-1): \qquad \boldsymbol{x}' = \begin{pmatrix} -3 & -2 \\ 2 & 2 \end{pmatrix} \boldsymbol{x}$$

$$(2,2): \qquad \boldsymbol{x}' = \begin{pmatrix} -3 & 4 \\ 4 & -4 \end{pmatrix} \boldsymbol{x}$$

$$(2,-2): \qquad \boldsymbol{x}' = \begin{pmatrix} -3 & 4 \\ 4 & 4 \end{pmatrix} \boldsymbol{x}$$

(ii) $x' = y \qquad y' = -x + x^3$

Again, the first task is to find the critical points. For $x' = 0$, we need to require that $y = 0$. Then, for $y' = 0$, we need $x = x^3$. This means $x = 0, \pm 1$. So our critical points are $(0,0)$, $(-1,0)$, and $(1,0)$.

The Jacobian matrix of the system at (x, y) is

$$\begin{pmatrix} 0 & 1 \\ -1 + 3x^2 & 0 \end{pmatrix},$$

so we end up with the following linearizations at each of the critical points.

$$(0,0): \qquad \boldsymbol{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{x}$$

$$(-1,0): \qquad \boldsymbol{x}' = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \boldsymbol{x}$$

$$(1,0): \qquad \boldsymbol{x}' = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \boldsymbol{x}$$

□

So, we've found the critical points and the linearizations of the original nonlinear system near them. Now what? Normally, we would go through our linear systems analysis to determine the type and stability of each critical point: remember, we can do this from just determining the eigenvalues. Ideally, the type and stability of the origin in these linear systems would correspond to the type and stability of the associated critical point in the nonlinear system.

Exercise. For each of the critical points in both systems given in Example 36.1, determine the type and stability of the origin in the linearized systems.

However, there's...

1. A Caveat, Or A Term's A Term No Matter How Small

In turning our nonlinear system near a critical point into the linear system 36.2, we disregarded terms of order two or higher on the grounds that they were very small. Is it really safe to do that? In principle, after all, those terms could influence the behavior of the system in ways that make this linearization method unreliable. It turns out that as long as linearized system's critical point is not one of several "borderline" cases, this doesn't affect the qualitative type of the critical point.

1.1. Simple Eigenvalues. If the linearized system near (x_0, y_0) has simple (*i.e.*, distinct) eigenvalues, the only problems occur when the linearization predicts we will have a center. Spirals, nodes, and saddles are all preserved. Moreover, stability is preserved: we can't go from having an unstable spiral in the linearized system to having a stable spiral in the nonlinear system. Thus if our linearization predicts we have a spiral, node, or saddle, we can conclude that the critical point will be of the same type.

Here's an example of how this fails when the linearization predicts a center.

EXAMPLE 36.2. *Show that the two systems*

$$x' = -y + x(x^2 + y^2) \qquad y' = x + y(x^2 + y^2)$$
$$x' = -y - x(x^2 + y^2) \qquad y' = x - y(x^2 + y^2)$$

have the same linearized systems at the critical point $(0, 0)$*, but different phase portraits.*

The Jacobian matrix of the first system is

$$\begin{pmatrix} 3x^2 + y^2 & -1 + 2xy \\ 1 + 2xy & x^2 + 3y^2 \end{pmatrix}.$$

The Jacobian matrix of the second system is

$$\begin{pmatrix} -3x^2 - y^2 & -1 - 2xy \\ 1 - 2xy & -x^2 - 3y^2 \end{pmatrix}.$$

At $(0, 0)$, then, both have the same linearization, namely

$$\boldsymbol{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{x}.$$

The eigenvalues of this matrix are $\pm i$, and so this linearized system has a center.

However, we can analyze this particular system more directly by using polar coordinates. If we do that, the two systems become

$$r' = r^3 \qquad \theta' = 1$$
$$r' = -r^3 \qquad \theta' = 1.$$

Thus, non-zero trajectories for the first system will expand, as $r' > 0$ for all $r > 0$, while non-zero trajectories for the second system will decay as $r' < 0$. We actually have an unstable spiral for the first system and an asymptotically stable spiral for the second system, even though the linearization predicted a center. These spirals, however, rotate in the same direction as the predicted center. □

The previous example demonstrated the sort of thing we would need to do if we actually wanted to determine the phase portrait near a predicted center. In general, this will require a more direct computation rather than a linearization. For our purposes, though, it will generally suffice to be aware that we can't trust the prediction of a center; even small higher order terms can throw a center to a spiral of either stability.

1.2. Repeated Eigenvalues. In the case of a repeated eigenvalue λ, we ended up getting either a star node or a degenerate node depending on whether λ was complete or defective. This is also a delicate case; we will, however, be able to conclude something.

If *lambda* is complete, and the linearization predicts a star node, we can conclude only that the nonlinear system will have a node (possibly degenerate or star) at the associated critical point. On the other hand, if λ is defective, we could get a node (again, it could possibly be degenerate or star) or even a spiral point. The stability of the linearized system, however, will be preserved.

Remark. Throughout this entire discussion of systems, we've assumed that our coefficient matrices are nonsingular; that is, they don't have any zero eigenvalues. This is an interesting case, though, since it's the only case where you get substantially different behavior between the linearization and the nonlinear system.