

LECTURE 3

Linear First Order Equations

Recall that an n th linear ordinary differential equation has the following form:

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

for some continuous functions $a_n(t), g(t)$ without restriction. In other words, a differential equation is linear if it looks like a polynomial, with differentiation instead of exponentiation.

This means that linear first order equations have a very simple form.

$$a_1(t)y'(t) + a_2(t)y(t) = g(t)$$

This isn't a very suitable form to work with, however. What we want to do is to divide through by $a_1(t)$ to obtain the following standard form:

$$(3.1) \quad \boxed{\frac{dy}{dt} + p(t)y(t) = q(t)}$$

REMARK. It is absolutely essential that the equation be in this form before we try to solve it. The method is completely dependant on the coefficient of y' being 1 and $p(t)$ having the correct sign.

Notice that if $p(t) = 0$ or $q(t) = 0$, Equation 3.1 is not only linear, but separable as well. However, if $p(t) \neq 0$ and $q(t) \neq 0$, separation of variables will not work. We'll need to be a bit more clever to solve this equation.

1. Solution Method: Integrating Factors

The key to solving just about any first order equation is to put it in a form where we can integrate both sides. There's no direct way to do that in this case, however. What we'll do is hazard a guess as to something that might help, plug in, and then figure out specifics. This will be a recurring theme in this class.

Let's suppose we multiply Equation 3.1 by some function $\mu(t)$.

$$(3.2) \quad \mu(t) \frac{dy}{dt} + \mu(t)p(t)y(t) = \mu(t)q(t)$$

The left hand side of Equation 3.2 will hopefully look familiar; it resembles the equation for the product rule from calculus.

$$(3.3) \quad y(t)\mu(t) + \mu'(t)y(t) = \frac{d}{dt} [\mu(t)y(t)]$$

If we could choose our function $\mu(t)$ such that this were the case, would be convenient, because then the left hand side of Equation 3.2 would just be the derivative of some product with respect to t . Since the right hand side is nothing more than some function of t , we could then integrate both sides and solve for $y(t)$.

So what does $\mu(t)$ need to be? For the left hand side of Equation 3.2 to be equivalent to the left hand side of Equation 3.3, we need

$$\mu'(t) = \mu(t)p(t).$$

This is a separable equation, and it's not hard to see that the solution is

$$(3.4) \quad \mu(t) = e^{\int p(t) dt}.$$

EXERCISE. Check the above assertion.

With this choice of $\mu(t)$, we can rewrite Equation 3.2 as

$$\frac{d}{dt} [\mu(t)y(t)] = \mu(t)q(t)$$

and solve for $y(t)$:

$$\begin{aligned} \mu(t)y(t) &= \int \mu(t)q(t) dt \\ y(t) &= \frac{\int \mu(t)q(t) dt}{\mu(t)}. \end{aligned}$$

REMARK. You may (and should!) notice that there is no constant of integration showing up on the left hand side despite us having integrated it. What we've implicitly done is to combine it with whatever constant of integration would come up when we compute $\int \mu(t)q(t) dt$, as you'll see in the examples to come. We've also done this with the constant that ought to appear when we find $\mu(t)$. It's imperative not to forget the constant at the final stage, however, or your answer will be very, very wrong.

This method is known as the *method of integrating factors* and $\mu(t) = e^{\int p(t) dt}$ as the integrating factor of Equation 3.1. My recommendation is not to memorize the last line in the solution, but rather to compute $\mu(t)$, multiply through, and go from there, which is what I will do in the examples below. Also, notice that we always get an explicit solution using this method.

Before we do some examples, let's summarize the previous discussion. The steps to solve a first order linear differential equation are:

- (1) Put the equation in the correct form

$$y' + p(t)y = q(t).$$

- (2) Calculate the integrating factor,

$$\mu(t) = e^{\int p(t) dt}.$$

- (3) Multiply both sides of the equation by $\mu(t)$.
- (4) Integrate both sides, being careful not to forget the constant of integration.
- (5) Solve for $y(t)$, using the initial condition (if applicable) to calculate the constant of integration.

2. Examples

EXAMPLE 3.1.

$$\cos(x)y' + \sin(x)y = 2 \cos^3(x) \sin(x) - 1 \quad y\left(\frac{\pi}{4}\right) = -\sqrt{2}$$

We first need to put this equation in the correct form, so divide through by $\cos(x)$.

$$y' + \tan(x)y = 2 \cos^2(x) \sin(x) - \sec(x)$$

Comparing this to Equation 3.1, we see that $p(x) = \tan(x)$, so our integrating factor is

$$\begin{aligned} \mu(x) &= e^{\int \tan(x) dx} \\ &= e^{\ln \sec(x)} \\ &= \sec(x), \end{aligned}$$

using the fact that $e^{\ln f(x)} = f(x)$. So, we multiply our differential equation through by $\sec(x)$ and solve for $y(x)$.

$$\begin{aligned}\sec(x)y' + \sec(x)\tan(x)y &= 2\cos(x)\sin(x) - \sec^2(x) \\ (\sec(x)y)' &= 2\cos(x)\sin(x) - \sec^2(x) \\ \sec(x)y(x) &= \int 2\cos(x)\sin(x) - \sec^2(x) dx \\ &= \int \sin(2x) - \sec^2(x) dx \\ &= -\frac{1}{2}\cos(2x) - \tan(x) + c \\ y(x) &= -\frac{1}{2}\cos(2x)\cos(x) - \sin(x) + c\cos(x).\end{aligned}$$

Now we use our initial condition to compute c .

$$\begin{aligned}-\sqrt{2} &= -\frac{1}{2}\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) + c\cos\left(\frac{\pi}{4}\right) \\ &= -\frac{\sqrt{2}}{2} + c\frac{\sqrt{2}}{2} \Rightarrow c = -1\end{aligned}$$

Thus our particular solution is

$$y(x) = -\frac{1}{2}\cos(2x)\cos(x) - \sin(x) - \cos(x)$$

□

EXAMPLE 3.2.

$$ty' + 2y = t^2 - t \quad y(1) = 1$$

Once again, we start by putting this into standard form, which requires division by t :

$$y' + \frac{2}{t}y = t - 1$$

Now we compute the integrating factor:

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln(t)} = e^{\ln(t^2)} = t^2$$

We continue by multiplying through by $\mu(t)$ and using the product rule.

$$\begin{aligned}(t^2y)' &= t^3 - t^2 \\ t^2y &= \int t^3 - t^2 dt \\ &= \frac{1}{4}t^4 - \frac{1}{3}t^3 + c \\ y(t) &= \frac{t^2}{4} - \frac{t}{3} + \frac{c}{t^2}.\end{aligned}$$

Using our initial condition to find c gives

$$1 = \frac{1}{4} - \frac{1}{3} + c \Rightarrow c = \frac{13}{12}$$

and our particular solution is

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{13}{12t^2}$$

□

EXAMPLE 3.3.

$$2y' - y = 4 \sin(3t) \quad y(0) = -1$$

First, divide by 2 to put this in the correct form.

$$y' - \frac{y}{2} = 2 \sin(3t)$$

Next, compute $\mu(t)$.

$$\mu(t) = e^{\int -\frac{1}{2} dt} = e^{-\frac{t}{2}}$$

Multiply through by $\mu(t)$ and write the left hand side as a product.

$$\left(e^{-\frac{t}{2}} y \right)' = 2e^{-\frac{t}{2}} \sin(3t)$$

Integrate both sides and solve for $y(t)$.

$$\begin{aligned} e^{-\frac{t}{2}} y(t) &= \int 2e^{-\frac{t}{2}} \sin(3t) dt \\ &= -\frac{24}{37} e^{-\frac{t}{2}} \cos(3t) - \frac{4}{37} e^{-\frac{t}{2}} \sin(3t) + c \\ y(t) &= -\frac{24}{37} \cos(3t) - \frac{4}{37} \sin(3t) + ce^{\frac{t}{2}} \end{aligned}$$

Finally, we compute the constant of integration.

$$-1 = -\frac{24}{37} + c \Rightarrow c = -\frac{13}{37}$$

Thus our particular solution is

$$y(t) = -\frac{1}{37} \left(24 \cos(3t) + 4 \sin(3t) + 13e^{\frac{t}{2}} \right)$$

□

EXAMPLE 3.4.

$$ty' - 2y = t^4 \sin(t) + t^3 - 3t^5 \quad y(\pi) = 2$$

First, divide by t to put this in the correct form.

$$y' - \frac{2}{t}y = t^3 \sin(t) + t^2 - 3t^4$$

Next, compute $\mu(t)$.

$$\mu(t) = e^{\int -\frac{2}{t} dt} = e^{-2 \ln(t)} = t^{-2}$$

Multiply through and write the left hand side as a product.

$$(t^{-2}y)' = t \sin(t) + 1 - 3t^2$$

Integrate both sides and solve for $y(t)$.

$$\begin{aligned} t^{-2}y &= \int t \sin(t) + 1 - 3t^2 dt \\ &= -t \cos(t) + \sin(t) + t - t^3 + c \\ y(t) &= t^3 \cos(t) + t^2 \sin(t) + t^3 - t^5 + ct^2 \end{aligned}$$

Plug in the initial condition to compute c .

$$\begin{aligned}2 &= \pi^3 \cos(\pi) + \pi^2 \sin(\pi) + \pi^3 - \pi^5 + c\pi^2 \\2 + \pi^5 &= c\pi^2 \\c &= \frac{2 + \pi^5}{\pi^2}\end{aligned}$$

Our particular solution is then

$$y(t) = t^3 \cos(t) + t^2 \sin(t) + t^3 - t^5 + \frac{2 + \pi^5}{\pi^2} t^2$$

□