LECTURE 28

IVPs with Laplace Transforms

Now that we have a good grasp of how to take Laplace transforms and inverse transforms, let's return to differential equations. First, we should recall the following formula, with $f^{(n)}$ denoting the nth derivative of f:

$$\mathcal{L}\left\{f^{(n)}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

We'll be dealing exclusively in this lecture with second order differential equations, so in particular, we'll need

$$\mathcal{L}\left\{y'\right\} = sY(s) - y(0)$$

and

$$\mathcal{L}\left\{y''\right\} = s^2 Y(s) - sy(0) - y'(0).$$

You should be familiar with the general formula, however.

Remark. Notice that we must have our initial conditions at $t = 0$ to use Laplace transforms.

Example 28.1. *Solve the following IVP using Laplace transforms.*

$$y'' - 5y' - 6y = 5t \qquad y(0) = -1 \quad y'(0) = 2$$

We begin by transforming both sides of the equation:

$$\begin{aligned}
\mathcal{L}\left\{y''\right\} - 5\mathcal{L}\left\{y'\right\} - 6\mathcal{L}\left\{y\right\} &= 5\mathcal{L}\left\{t\right\} \\
s^2 Y(s) - sy(0) - y'(0) - 5sY(s) + 5y(0) - 6Y(s) &= \frac{5}{s^2} \\
\left(s^2 - 5s - 6\right) Y(s) + s - 2 - 5 &= \frac{5}{s^2}.
\end{aligned}$$

As we've already begun doing, now we solve for $Y(s)$.

$$\begin{aligned}
Y(s) &= \frac{5}{s^2(s^2 - 5s - 6)} + \frac{7 - s}{s^2 - 5s - 6} \\
&= \frac{5}{s^2(s-6)(s+1)} + \frac{7 - s}{(s-6)(s+1)} \\
&= \frac{5 + 7s^2 - s^3}{s^2(s-6)(s+1)}
\end{aligned}$$

We now have an expression for $Y(s)$, which is the Laplace transform of the solution $y(t)$ to the initial value problem. We've simplified it as much as we can; now it's time to take the inverse transform. The partial fraction decomposition is

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-6} + \frac{D}{s+1}.$$

Setting numerators equal gives us

$$6 + 7s^2 - s^3 = As(s-6)(s+1) + B(s-6)(s+1) + Cs^2(s+1) + Ds^2(s-6).$$

We can find constants by choosing key values of s.

$$\begin{aligned}
s=0: &\quad 6=-6B &\Rightarrow&\quad B=-1\\
s=6: &\quad 42=252C &\Rightarrow&\quad C=\frac{1}{6}\\
s=-1: &\quad 14=-7D &\Rightarrow&\quad D=-\frac{1}{2}\\
s=1: &\quad 12=-10A+\frac{77}{6} &\Rightarrow&\quad A=\frac{1}{12}
\end{aligned}$$

So

$$\begin{aligned}
Y(s) &= \frac{1}{12}\frac{1}{s}-\frac{1}{s^2}+\frac{1}{6}\frac{1}{s-6}-\frac{1}{2}\frac{1}{s+1}\\
y(t) &= \frac{1}{12}-t+\frac{1}{6}e^{6t}-\frac{1}{2}e^{-t}.
\end{aligned}$$

□

Exercise. Solve the initial value problem in the previous example using Undetermined Coefficients. Do you get the same thing? Which method took less work?

Example 28.2. *Solve the following initial value problem.*

$$y''+2y'+5y=\cos(t)-10t \qquad y(0)=0 \quad y'(0)=1$$

We begin by transforming the entire equation and solving for $Y(s)$.

$$\begin{aligned}
\mathcal{L}\left\{y''\right\}+2\mathcal{L}\left\{y'\right\}+5\mathcal{L}\left\{y\right\} &= \mathcal{L}\left\{\cos(t)\right\}-10\mathcal{L}\left\{t\right\}\\
s^2Y(s)-sy(0)-y'(0)+2(sY(s)-y(0))+5Y(s) &= \frac{s}{s^2+1}-\frac{10}{s^2}\\
\left(s^2+2s+5\right)Y(s)-1 &= \frac{s}{s^2+1}-\frac{10}{s^2}
\end{aligned}$$

So we have

$$\begin{aligned}
Y(s) &= \frac{s}{(s^2+1)(s^2+2s+5)}-\frac{10}{s^2(s^2+2s+5)}+\frac{1}{s^2+2s+5}\\
&= Y_1(s)+Y_2(s)+Y_3(s).
\end{aligned}$$

Now we'll have to take inverse transforms. This will require doing partial fractions on the first two pieces.

Let's start with the first one.

$$Y_1(s)=\frac{s}{(s^2+1)(s^2+2s+5)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+2s+5}$$

After putting everything over a common denominator, we set numerators equal.

$$\begin{aligned}
s &= As(s^2+2s+5)+B(s^2+2s+5)+Cs(s^2+1)+D(s^2+1)\\
&= (A+C)s^3+(2A+B+D)s^2+(5A+2B+C)s+(5B+D)
\end{aligned}$$

This gives us the following system of equations, which we solve.

$$\begin{aligned}
A+C&=0\\
2A+B+D&=0\\
5A+2B+C&=1\\
5B+D&=0
\end{aligned}
\qquad\Rightarrow\qquad A=\frac{1}{5}\quad B=\frac{1}{10}\quad C=-\frac{1}{5}\quad D=-\frac{1}{2}$$

Thus our first term becomes

$$Y_1(s) = \frac{1}{5}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} - \frac{1}{5}\frac{s}{s^2+2s+5} - \frac{1}{2}\frac{1}{s^2+2s+5}.$$

We'll hold off on taking the inverse transform for the time being.

Now, let's deal with $Y_2(s)$.

$$Y_2(s) = -\frac{10}{s^2(s^2+2s+5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+2s+5}$$

We put everything over a common denominator and set numerators equal.

$$\begin{aligned} -10 &= As(s^2+2s+5) + B(s^2+2s+5) + Cs^3 + Ds^2 \\ &= (A+C)s^3 + (2A+B+D)s^2 + (5A+2B)s + 5B \end{aligned}$$

This gives the following system of equations.

$$\begin{aligned} A + C &= 0 \\ 2A + B + D &= 0 \\ 5A + 2B &= 0 \\ 5B &= -10 \end{aligned} \qquad \Rightarrow \qquad A = \frac{4}{5} \quad B = -2 \quad C = -\frac{4}{5} \quad D = \frac{2}{5}$$

Thus we have

$$Y_2(s) = \frac{4}{5}\frac{1}{s} - \frac{2}{s^2} - \frac{4}{5}\frac{s}{s^2+2s+5} + \frac{2}{5}\frac{1}{s^2+2s+5}.$$

Let's return to our original function.

$$\begin{aligned} Y(s) &= Y_1(s) + Y_2(s) + Y_3(s) \\ &= \frac{1}{5}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} + \frac{4}{5}\frac{1}{s} - \frac{2}{s^2} + \left(-\frac{1}{5} - \frac{4}{5}\right)\frac{s}{s^2+2s+5} + \left(-\frac{1}{2} + \frac{2}{5} + 1\right)\frac{1}{s^2+2s+5} \\ &= \frac{1}{5}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} + \frac{4}{5}\frac{1}{s} - \frac{2}{s^2} - \frac{s}{(s+1)^2+4} + \frac{9}{10}\frac{1}{(s+1)^2+4} \end{aligned}$$

Now we have to adjust the last two terms to make them suitable for the inverse transform. Namely, we need to have $s+1$ in the numerator of the second to last, and 2 in the numerator of the last.

$$\begin{aligned} &= \frac{1}{5}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} + \frac{4}{5}\frac{1}{s} - \frac{2}{s^2} - \frac{s+1-1}{(s+1)^2+4} + \frac{9}{10}\frac{1}{(s+1)^2+4} \\ &= \frac{1}{5}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} + \frac{4}{5}\frac{1}{s} - \frac{2}{s^2} - \frac{s+1}{(s+1)^2+4} + \frac{19}{10}\frac{1}{(s+1)^2+4} \\ &= \frac{1}{5}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} + \frac{4}{5}\frac{1}{s} - \frac{2}{s^2} - \frac{s+1}{(s+1)^2+4} + \frac{19}{20}\frac{2}{(s+1)^2+4} \end{aligned}$$

So our solution is

$$y(t) = \frac{1}{5}\cos(t) + \frac{1}{10}\sin(t) + \frac{4}{5} - 2t - e^{-t}\cos(2t) + \frac{19}{20}e^{-t}\sin(2t).$$

□

We could have done both of the preceding examples using Undetermined Coefficients. In fact, it would have been a lot less work. Let's do some involving step functions, which is where Laplace transforms really shine.

EXAMPLE 28.3. *Solve the following initial value problem.*

$$y'' - 5y' + 6y = 2 - u_2(t)e^{2t-4} \qquad y(0) = 0 \quad y'(0) = 0$$

As before, we begin by transforming everything. Before we do that, however, we need to write the coefficient function of $u_2(t)$ as a function evaluated at $t-2$.

$$y'' - 5y' + 6y = 2 - u_2(t)e^{2(t-2)}$$

Now we can transform.

$$\begin{aligned} \mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} &= 2\mathcal{L}\{1\} - \mathcal{L}\left\{u_2(t)e^{2(t-2)}\right\} \\ s^2Y(s) - sy(0) - y'(0) - 5sY(s) + 5y(0) - 6Y(s) &= \frac{2}{s} - e^{-2s}\mathcal{L}\left\{e^{2t}\right\} \\ (s^2 - 5s + 6)Y(s) &= \frac{2}{s} - e^{-2s}\frac{1}{s-2} \end{aligned}$$

So we end up with

$$\begin{aligned} Y(s) &= \frac{2}{s(s-3)(s-2)} - e^{-2s}\frac{1}{(s-3)(s-2)^2} \\ &= Y_1(s) + e^{-2s}Y_2(s). \end{aligned}$$

Since one of these terms has an exponential, we'll need to deal with them separately. I'll leave it to you to check all of the partial fractions.

$$\begin{aligned} Y_1(s) &= \frac{1}{3}\frac{1}{s} + \frac{2}{3}\frac{1}{s-3} - \frac{1}{s-2} \\ Y_2(s) &= -\frac{1}{s-3} + \frac{1}{s-2} + \frac{1}{(s-2)^2} \end{aligned}$$

Thus we have

$$Y(s) = \frac{1}{2}\frac{1}{s} + \frac{2}{3}\frac{1}{s-3} - \frac{1}{s-2} + e^{-2s}\left(-\frac{1}{s-3} + \frac{1}{s-2} + \frac{1}{(s-2)^2}\right)$$

and

$$\begin{aligned} y(t) &= \frac{1}{2} + \frac{2}{3}e^{3t} - e^{2t} + u_2(t)\left(-e^{3(t-2)} + e^{2(t-2)} + (t-2)e^{2(t-2)}\right) \\ &= \frac{1}{2} + \frac{2}{3}e^{3t} - e^{2t} + u_2(t)\left(-e^{3t-6} - e^{2t-4} + te^{2t-4}\right) \end{aligned}$$

once we observe that $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = te^{at}$. □

EXAMPLE 28.4. *Solve the following initial value problem.*

$$y'' + 4y = 8 + tu_4(t) \qquad y(0) = 0 \quad y'(0) = 0$$

We need to first write the coefficient function of $u_4(t)$ in the form $h(t-4)$ for some function $h(t)$. So we write $h(t-4) = t = t - 4 + 4$ and conclude $h(t) = t + 4$. So our equation is

$$y'' + 4y = 8 + ((t-4) + 4)u_4(t).$$

Now, we want to Laplace transform everything.

$$\begin{aligned} \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} &= 8\mathcal{L}\{1\} + \mathcal{L}\{((t-4)+4)u_4(t)\} \\ s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= \frac{8}{s} + e^{-4s}\mathcal{L}\{t+4\} \\ (s^2 + 4)Y(s) &= \frac{8}{s} + e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) \end{aligned}$$

So we have

$$\begin{aligned} Y(s) &= \frac{8}{s(s^2+4)} + e^{-4s}\left(\frac{1}{s^2(s^2+4)} + \frac{4}{s(s^2+4)}\right) \\ &= \frac{8}{s(s^2+4)} + e^{-4s}\frac{1+4s}{s^2(s^2+4)} \qquad = Y_1(s) + e^{-4s}Y_2(s), \end{aligned}$$

where we've consolidated the two fractions being multiplied by the exponential to reduce the number of partial fraction decompositions we need to compute. After doing partial fractions (leaving the details for you to check), we have

$$Y_1(s) = \frac{2}{s} - \frac{2s}{s^2+4}$$

and

$$Y_2(s) = \frac{1}{s} + \frac{1}{4}\frac{1}{s^2} - \frac{s}{s^2+4} - \frac{1}{4}\frac{1}{s^2+4},$$

so

$$\begin{aligned} Y(s) &= \frac{2}{s} - \frac{2s}{s^2+4} + e^{-4s}\left(\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} - \frac{s}{s^2+4} - \frac{1}{4}\frac{1}{s^2+4}\right) \\ &= \frac{2}{s} - 2\frac{s}{s^2+4} + e^{-4s}\left(\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} - \frac{s}{s^2+4} - \frac{1}{8}\frac{2}{s^2+4}\right) \end{aligned}$$

and the solution is

$$\begin{aligned} y(t) &= 2 - 2\cos(2t) + u_4(t)\left(1 + \frac{1}{4}(t-4) - \cos(2(t-4)) - \frac{1}{8}\sin(2(t-4))\right) \\ &= 2 - 2\cos(2t) + u_4(t)\left(\frac{1}{4}t - \cos(2t-8) - \frac{1}{8}\sin(2t-8)\right). \end{aligned}$$

□