LECTURE 24

Laplace Transforms

One approach to solving differential equations is to tackle them directly, which is what we've been doing. Sometimes, however, it's convenient to transform the problem into a different one, which we can then solve more easily and transform back. One example of such a transformation is the Laplace transform, which we'll discuss now. In some cases, it lets us more easily solve problems we could solve with other methods, while in others, it lets us deal with problems we have no other method to solve.

For the basic examples, such as the sorts of linear homogeneous equations with constant coefficients that we looked at earlier, we'll find that the Laplace transform is more messy than we would otherwise require. When we start having some basic nonhomogeneous equations, we could still use other methods, but the amount of work is generally a wash. When our forcing functions start to get more complicated, however, Laplace transforms are a great method to have at hand.

1. The Definition

There is one basic notion we have to get out of the way before we can define the Laplace transform.

DEFINITION 24.1. A function f is called *piecewise continuous* on an interval $[a, b]$ if $[a, b]$ can be broken into a finite number of subintervals $[a_n, b_n]$ such that f is continuous on each open subinterval (a_n, b_n) and has a finite limit at every endpoint a_n, b_n.

In other words, a piecewise continuous function has only a finite number of "jumps" and doesn't have any asymptotes where it blows up to plus or minus infinity. There are many examples of these; we'll see several down the road.

Now, we can define the Laplace transform of a function.

DEFINITION 24.2. Suppose that $f(t)$ is a piecewise continuous function. The Laplace transform of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$, is given by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt. \tag{24.1}$$

REMARK. There is an alternate notation for the Laplace transform that we will commonly use. Notice that the definition of $\mathcal{L}\{f(t)\}$ introduces a new variable, s, then is a definite integral with respect to t. As a result, computing this transform yields a function which depends on s. Thus, we will use the notation

$$\mathcal{L}\{f(t)\} = F(s).$$

It should also be noted that the integral in the definition of $\mathcal{L}\{f(t)\}$ is an improper integral. In our first examples of computing Laplace transforms, we'll review how these work.

EXAMPLE 24.3. *Compute* $\mathcal{L}\{1\}$. Plugging $f(t) = 1$ into the definition (24.1), we have

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt.$$

Recall that to calculate this improper integral, we need to convert it to a limit as follows.

$$\begin{aligned} &= \lim_{N\to\infty} \int_0^N e^{-st}\,dt \\ &= \lim_{N\to\infty} \left[-\frac{1}{s}e^{-st}\right]_0^N \\ &= \lim_{N\to\infty} \left(-\frac{1}{s}e^{-Ns} + \frac{1}{s}\right) \end{aligned}$$

At this point we need to be careful: the value of s will affect our answer. If $s < 0$, the exponent of our exponential is positive, so the limit in question will diverge as the exponential goes to infinity. On the other hand, if $s > 0$, the exponential will go to 0 and the limit will converge.

Thus, we restrict our attention to the case where $s > 0$ and conclude that

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \text{for } s > 0.$$

□

Notice that we had to put a restriction on the domain of our Laplace transform. This will always be the case: these integrals will not always converge for any s. At the moment, we can brush this to the side as a technical detail, but it's important to keep in mind that Laplace transforms are not defined for all s.

EXAMPLE 24.4. *Compute $\mathcal{L}\{e^{at}\}$ for $a \neq 0$.*
By definition,

$$\begin{aligned} \mathcal{L}\left\{e^{at}\right\} &= \int_0^\infty e^{-st}e^{at}\,dt \\ &= \int_0^\infty e^{(a-s)t}\,dt \\ &= \lim_{N\to\infty}\left[\frac{1}{a-s}e^{(a-s)t}\right]_0^N \\ &= \lim_{N\to\infty}\left(\frac{1}{a-s}e^{(a-s)N} - \frac{1}{a-s}\right) \\ &= \frac{1}{s-a} \qquad \text{for } s > a. \end{aligned}$$

□

EXAMPLE 24.5. *Compute $\mathcal{L}\{\sin(at)\}$.*

$$\begin{aligned} \mathcal{L}\{\sin(at)\} &= \int_0^\infty e^{-st}\sin(at)\,dt \\ &= \lim_{N\to\infty}\int_0^N e^{-st}\sin(at)\,dt. \end{aligned}$$

Integration by parts yields

$$= \lim_{N\to\infty}\left(\frac{1}{a}\left(1 - e^{-sn}\cos(an)\right) - \frac{s}{a}\left(\frac{1}{a}e^{-sn}\sin(an) + \frac{s}{a}\int_0^n e^{-st}\sin(at)\,dt\right)\right).$$

Doing a bit more jiggling, we get

$$F(s) = \frac{1}{a} - \frac{s}{a}F(s)$$
$$\mathcal{L}\left\{\sin(at)\right\} = F(s) = \frac{a}{s^2+a^2} \qquad \text{provided } s > 0.$$

□

Example 24.6. *If $f(t)$ is a piecewise continuous function with piecewise continuous derivative $f'(t)$, express $\mathcal{L}\left\{f'(t)\right\}$ in terms of $\mathcal{L}\left\{f(t)\right\}$.*

We plug f' into the definition (24.1).

$$\begin{aligned}\mathcal{L}\left\{f'\right\} &= \int_0^\infty e^{-st} f'\, dt \\ &= \lim_{N\to\infty} \int_0^N e^{-st} f'\, dt\end{aligned}$$

The next step is to integrate by parts.

$$\begin{aligned}&= \lim_{N\to\infty} \left(e^{-st} f \Big|_0^N + s\int_0^N e^{-st} f\, dt \right) \\ &= \lim_{N\to\infty} e^{-sN} f(N) - f(0) + s\int_0^\infty e^{-st} f\, dt \\ &= s\mathcal{L}\left\{f(t)\right\} - f(0) \qquad \text{provided } s > 0\end{aligned}$$

□

Doing this repeatedly, one can find

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n \mathcal{L}\left\{f(t)\right\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Example 24.7. *If $f(t)$ is a piecewise continuous function, express $\mathcal{L}\left\{e^{at}f(t)\right\}$ in terms of $\mathcal{L}\left\{f(t)\right\}$.*

We begin by plugging into (24.1).

$$\begin{aligned}\mathcal{L}\left\{e^{at}f(t)\right\} &= \int_0^\infty e^{-st}e^{at} f(t)\, dt \\ &= \int_0^\infty e^{(a-s)t} f(t)\, dt\end{aligned}$$

This looks rather like the definition of $\mathcal{L}\left\{f(t)\right\} = F(s)$, but it's not quite, since the exponent is $a - s$. However, if we substitute $u = s - a$, we get the following.

$$\begin{aligned}&= \int_0^\infty e^{-ut} f(t)\, dt \\ &= F(u) \\ &= F(s-a).\end{aligned}$$

Thus if we take the Laplace transform of a function multiplied by e^{at}, we'll get the Laplace transform of the original function shifted by a. This will be useful to keep in mind. □

2. Laplace Transforms

In general, we won't be computing our Laplace transforms from scratch; we'll be using a table. The table doesn't include every Laplace transform we might encounter, but it does have the ones we will commonly see. My recommendation is to know the transforms you see coming up all the time, and use the table as insurance or if you encounter one you're not familiar with. From now on, the examples we work will be done with a thought process that supposes we have a table in front of us, even if we don't.

There is one important fact we need to get out of the way: the Laplace transform is linear.

THEOREM 24.1. *Given piecewise continuous functions $f(t)$ and $g(t)$,*

$$\mathcal{L}\left\{af(t)+bg(t)\right\}=a\mathcal{L}\left\{f(t)\right\}+b\mathcal{L}\left\{g(t)\right\}$$

for any constants a, b.

This follows from the linearity of integration.

From a practical perspective, this is great. It means that we don't have to worry about constants or sums; we can just decompose our function into individual pieces, transform them, and then put everything back together. Let's do a few examples.

EXAMPLE 24.8. *Find the Laplace transforms of the following functions.*

(i) $f(t)=6e^{-5t}+e^{3t}+5t^3-9$

$$\begin{aligned} F(s)=\mathcal{L}\left\{f(t)\right\} &= 6\mathcal{L}\left\{e^{-5t}\right\}+\mathcal{L}\left\{e^{3t}\right\}+5\mathcal{L}\left\{t^3\right\}-9\mathcal{L}\left\{1\right\} \\ &= 6\frac{1}{s-(-5)}+\frac{1}{s-3}+5\frac{3!}{s^{3+1}}-9\frac{1}{s} \\ &= \frac{6}{s+5}+\frac{1}{s-3}+\frac{30}{s^4}-\frac{9}{s} \end{aligned}$$

(ii) $g(t)=4\cos(4t)-2\sin(4t)-3\cos(8t)$

$$\begin{aligned} G(s)=\mathcal{L}\left\{g(t)\right\} &= 4\mathcal{L}\left\{\cos(4t)\right\}-2\mathcal{L}\left\{\sin(4t)\right\}-3\mathcal{L}\left\{\cos(10t)\right\} \\ &= 4\frac{s}{s^2+4^2}-2\frac{4}{s^2+4^2}-3\frac{s}{s^2+10^2} \\ &= \frac{4s-8}{s^2+16}-\frac{3s}{s^2+100} \end{aligned}$$

(iii) $h(t)=e^{2t}+\cos(3t)-e^{2t}\cos(3t)$

$$\begin{aligned} H(t)=\mathcal{L}\left\{h(t)\right\} &= \mathcal{L}\left\{e^{2t}\right\}+\mathcal{L}\left\{\cos(3t)\right\}-\mathcal{L}\left\{e^{2t}\cos(3t)\right\} \\ &= \frac{1}{s-2}+\frac{s}{s^2+3^2}-\frac{s-2}{(s-2)^2+3^2} \\ &= \frac{1}{s-2}+\frac{2}{s^2+9}-\frac{s-2}{(s-2)^2+9} \end{aligned}$$

□

3. Initial Value Problems

As interesting as that was, we're not interested in playing with Laplace transforms for their own sake. After all, this is a differential equations class. The reason we're discussing Laplace transforms is that they will help us solve certain initial value problems. Now that we know what the Laplace transform is, let's look at an example to see how this works and figure out what we have left to learn about Laplace transforms before we can apply them.

EXAMPLE 24.9. *Solve the following initial value problem using Laplace transforms.*

$$y'' - 6y' + 5y = 7t \qquad y(0) = -1 \qquad y'(0) = 2.$$

The first step in using Laplace transforms to solve an initial value problem is to transform both sides of the equation.

$$\begin{aligned} \mathcal{L}\left\{y''\right\} - 6\mathcal{L}\left\{y'\right\} + 5\mathcal{L}\left\{y\right\} &= 7\mathcal{L}\left\{t\right\} \\ s^2Y(s) - sy(0) - y'(0) - 6\left(sY(s) - y(0)\right) + 5Y(s) &= \frac{7}{s^2} \\ s^2Y(s) + s - 2 - 6\left(sY(s) + 1\right) + 5Y(s) &= \frac{7}{s^2} \end{aligned}$$

Next, we solve for $Y(s)$.

$$\begin{aligned} \left(s^2 - 6s + 5\right)Y(s) + s - 8 &= \frac{7}{s^2} \\ Y(s) &= \frac{7}{s^2\left(s^2 - 6s + 5\right)} + \frac{8 - s}{s^2 - 6s + 5} \end{aligned}$$

Now what? We want to solve for $y(t)$, but we have an expression for $Y(s) = \mathcal{L}\left\{y(t)\right\}$. Thus, to able to actually finish solving this problem, we'll have to discuss how to go backwards; we'll need to learn how to take inverse Laplace transforms.

4. Inverse Transforms

In this section, we'll be starting with the transform $F(s)$ and trying to find the original function $f(t)$. This is a slightly more complicated process than taking transforms, which was quite straightforward. We refer to $f(t)$ as the *inverse Laplace transform* of $F(s)$ and use the notation

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}.$$

Our starting point is that the inverse Laplace transform is linear, just like the original transform was.

THEOREM 24.2. *Given two Laplace transforms $F(s)$ and $G(s)$,*

$$\mathcal{L}^{-1}\left\{aF(s) + bG(s)\right\} = a\mathcal{L}^{-1}\left\{F(s)\right\} + b\mathcal{L}^{-1}\left\{G(s)\right\}$$

for any constants a, b.

So, we'll decompose our original transformed function into pieces, inverse transform, and then put everything back together.

This is where familiarity with the basic Laplace transforms and the table becomes handy. The key is to try to identify the desired inverse transform by looking at the denominator. For the most part, this will tell us what the original function will have to be, but occasionally, we will have to look at the numerator to distinguish between two potential inverses (*e.g.*, the denominators for the transforms of $\sin(at)$ and $\cos(at)$ are the same, but the numerators differ). Then, we know precisely how we have to write our function $F(s)$ so that it is the inverse transform of the function we've identified as the inverse. Sometimes this requires a little bit of algebra or arithmetic.

Let's look at some examples.

EXAMPLE 24.10. *Find the inverse transforms of the following.*

(i) $F(s) = \dfrac{6}{s} - \dfrac{1}{s-8} + \dfrac{4}{s-3}$

This one is quite straightforward. The denominator of the first term, $\frac{6}{s}$, indicates that this will be the Laplace transform of 1. Since $\mathcal{L}\{1\} = \frac{1}{s}$, we'll factor out the 6 before taking the inverse transform. For the second term, this is just the Laplace transform of e^{8t}, and there's nothing else to do with it. The third term is also an exponential, e^{3t}, and we'll need to factor out the 4 in the numerator before we inverse transform.

So we have

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= 6\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-8}\right\} + 4\mathcal{L}^{-1}\{1\}\, s-3 \\ f(t) &= 6(1) - e^{8t} + 4\left(e^{3t}\right) \\ &= 6 - e^{8t} + 4e^{3t}.\end{aligned}$$

We usually won't put quite so much detail into these.

(ii) $G(s) = \dfrac{12}{s+3} - \dfrac{1}{2s-4} + \dfrac{2}{s^4}$.

The first term is just the transform of e^{-3t} multiplied by 12, which we'll factor out before inverse transforming.

The second term looks like it *ought* to be an exponential, but it's got a $2s$ instead of an s in the denominator, and transforms of exponentials should just have s. We can fix this by factoring a 2 out of the denominator and then taking the inverse transform.

The third term has s^4 as its denominator. This indicates that it will be related to the transform of t^3. The numerator isn't quite correct, though, since $\mathcal{L}\left\{t^3\right\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$. So we would need the numerator to be 6, and right now it's 2. How do we fix this? We'll multiply by $\frac{3}{3}$, absorb the top 3 into the transform, and keep the $\frac{1}{3}$ out front.

Let's start by rewriting the transform, with these fixes incorporated.

$$\begin{aligned}G(s) &= 12\frac{1}{s-(-3)} - \frac{1}{2(s-2)} + \frac{3}{3}\frac{2}{s^4} \\ &= 12\frac{1}{s-(-3)} - \frac{1}{2}\frac{1}{s-2} + \frac{1}{3}\frac{6}{s^4}\end{aligned}$$

Now we can take the inverse transform.

$$g(t) = 12e^{-3t} - \frac{1}{2}e^{2t} + \frac{1}{3}t^3$$

(iii) $H(s) = \dfrac{4s}{s^2+25} + \dfrac{3}{s^2+16}$

The denominator of the first term, s^2+25, indicates that this should be the transform of either $\sin(5t)$ or $\cos(5t)$. The numerator is $4s$, though, which tells us that once we factor out the 4, it will be the transform of $\cos(5t)$.

The second term's denominator is s^2+16, so it will be the transform of either $\sin(4t)$ or $\cos(4t)$. The numerator is a constant, 3, so it will be the transform of $\sin(4t)$. The only problem is that the numerator of $\mathcal{L}\{\sin(4t)\}$ should be 4, while here it is 3. We'll fix this, as in the previous example, by multiplying by $\frac{4}{4}$.

We rewrite the transform.

$$\begin{aligned}H(s) &= 4\frac{1}{s^2+(5)^2} + \frac{4}{4}\frac{3}{s^2+4^2} \\ &= 4\frac{1}{s^2+5^2} + \frac{3}{4}\frac{4}{s^2+4^2}\end{aligned}$$

Then we take the inverse.

$$h(t) = 4\cos(5t) + \frac{3}{4}\sin(4t).$$

□