

LECTURE 1

Introduction

1. What Is A Differential Equation?

A *differential equation* is an equation that contains derivatives of a function. On one hand, this seems obvious. On the other, it's a remarkably subtle idea.

Why? Let's think back to algebra. Equations there were of the form

$$f(x) = c$$

for some known function $f(x)$ and some constant c . $f(x)$ may have been the derivative of another function, but we'd explicitly calculated it...the function was never unknown. Solving the equation meant finding values of the independent variable x that caused the function to have the desired value. So far, so good; this is old news. But what does that have to do with anything at the moment?

Here's the difference: the unknowns in a differential equation are *functions*, not numbers. This is an important conceptual jump, the central one in this course. We don't plug in a value for x and see what $f(x)$ evaluates to; we plug in expressions for a function and see what the whole mess simplifies to.

Another way of putting it (which you don't have to especially worry about, but which can be helpful) is that a differential equation has the form

$$L[y] = g(x)$$

where $y(x)$ is an unknown function (a variable in this case), $g(x)$ is some other (known) function not involving y , but rather only the independent variable, and L is a function defined on functions (what we in mathematics call an "operator").

An example of an operator is just $\frac{d}{dx}$. If you plug in a function y , you get $\frac{dy}{dx}$, which is just another function.

This has several consequences. First, it means that differential equations are hard. For this class, by "hard," I mean "they take some getting used to." We're used to our equations having numerical solutions, and we understand exactly what an equation says when we look at it. Now, we'll need to learn how to allow ourselves to abstract our notion of "equation." In the real world, differential equations are actually difficult. Most can't be solved, and some that can be solved require computers to do so.

Second, differential equations, when they have solutions, have infinitely many. Why? Well, let's look at a differential equation you've all seen before.

$$\frac{dy}{dx} = x \tag{1.1}$$

This is just an equation from calculus. What we want to find is a function $y(x)$ that, when differentiated at a point x gives us just x . We know how to solve this: just integrate.

$$y(x) = \frac{x^2}{2} + c \tag{1.2}$$

Since differentiation ignores constants, when we integrate we have to account for this by adding in the constant of integration c . This is where the infinity of solutions comes from, and this will come up every time we solve a differential equation. A solution of the form 1.2 is called a *general solution* because it can express every possible solution of the differential equation.

Usually, though, we're interested in solving a particular *initial value problem*. Initial value problems are a combination of a differential equation and some *initial conditions* which fix a specific, or *particular solution*. They have the form

$$y^{(k)}(t_0) = y_k$$

for whichever derivatives of y are necessary. This is no different from an integration problem like Equation 1.1; we fixed a particular value of y to determine c , and that gave us a unique function. So, when we are faced with an initial value problem, we're doing two things: we're finding all functions that satisfy the equation and then choosing the particular one that satisfies the given initial conditions.

Let's say we've got a differential equation. A function y is a solution to the differential equation if, when we take the relevant derivatives and plug them in, the equation "works": everything simplifies the way it should.

EXAMPLE 1.1. Consider

$$2x^2y'' - 3xy' - 3y = 0.$$

I claim $y(x) = x^{-\frac{1}{2}}$ is a solution to this equation. To check this, we'll need the first and second derivatives:

$$y'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$y''(x) = \frac{3}{4}x^{-\frac{5}{2}}$$

Then:

$$\begin{aligned} 2x^2 \left(\frac{3}{4}x^{-\frac{5}{2}} \right) - 3x \left(-\frac{1}{2}x^{-\frac{3}{2}} \right) - 3 \left(x^{-\frac{1}{2}} \right) &= 0 \\ \frac{3}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}} - 3x^{-\frac{1}{2}} &= 0 \\ 0 &= 0 \end{aligned}$$

So $y(x) = x^{-\frac{1}{2}}$ is in fact a solution. Here are some others:

$$y(x) = x^3,$$

$$y(x) = 12x^{-\frac{1}{2}},$$

$$y(x) = 6x^{-\frac{1}{2}} - 2x^3.$$

In fact, every solution to this differential equation is of the form

$$y(x) = c_1x^{-\frac{1}{2}} + c_2x^3 \tag{1.3}$$

for some constants c_1 and c_2 . So Equation 1.3 is the general solution to the differential equation, and we could pick out a particular solution if we specified some initial conditions. \square

We refer to a solution such as in Example 1.1 as an *explicit solution* since it is of the form $y = f(x)$; in other words, the only place y appears is on the left hand side and it is just by itself,

raised to the first power. We'll see examples soon where we won't be able to solve for y . In those cases, we will have to be content to find *implicit solutions*.

EXAMPLE 1.2. $y^2 = t^2 - 3$ is an implicit solution to

$$y' = \frac{t}{y} \quad y(-2) = 1$$

□

EXERCISE. Check that, indeed, $y^2 = t^2 - 3$ is a solution to the initial value problem given in Example 1.2.

EXAMPLE 1.3. Let's find the explicit solution to the problem given in Example 1.2. We know that $y^2 = t^2 - 3$ is a solution; now we want to solve for y . Doing so yields $y = \pm\sqrt{t^2 - 3}$. This is a problem: we want one expression for y , but here we have two. To figure out which one we want, we use the initial condition $y(-2) = 1$. Only one of the two possibilities will satisfy the initial condition. Plugging in, we see that the positive one is correct, and

$$y(t) = \sqrt{t^2 - 3}$$

is the explicit solution to this initial value problem. □

Checking that things are solutions is very important, especially early on. It's a good habit to get into, because it's very easy to make a sloppy mistake (such as a sign error) that can drastically change the solution.

The other issue we need to keep in mind is that solutions to differential equations are generally only valid on certain intervals. Consider Example 1.1. We know $y(x) = x^{-\frac{1}{2}}$ is a solution to this equation. . . but for what values of x ? It can't be all of them, since $y(x)$ isn't even defined at $x = 0$. It turns out that this solution is valid for $x > 0$. In fact, looking at Equation 1.3, no solution to this differential equation exists at $x = 0$! So even though, symbolically, $y(x)$ satisfied the differential equation for all x , the solution implied a certain restriction on the independent variable.

This is generally going to be the case: it's very rare when a solution to an initial value problem works for all values of the independent variable. The interval on which a solution is valid is called the *interval of validity*, and in some cases we'll talk about being able to deduce some information about the interval of validity of a solution without actually having to find the solution.

2. Why do we care?

Good question (and one that you should be earnestly asking throughout life)! We care because differential equations can be used to model all sorts of continuous processes, such as radioactive decay, continuous interest, mixing problems, harmonic motion, pendulums, heat diffusion, and the movement of a string. Thus, from the perspective of a physicist or an engineer, differential equations are a foundational tool. Most of the math you will see further down the road in your major courses will involve differential equations.

Even if that weren't the case, differential equations are a rich and interesting area. There's a lot of interesting and unexpected behavior that can occur that we will unfortunately only be able to begin looking at.

3. Examples

Now let's look at some examples of differential equations.

EXAMPLE 1.4 (Radioactive Decay). Suppose we have a certain number of atoms $N(t)$ of some radioactive isotope, let's say Carbon-14 (C-14). All such isotopes have a certain decay constant λ associated to them which gives the rate at which a quantity of these atoms will decay. In the case of C-14, $\lambda = 1.21 \times 10^4 \text{ yr}^{-1}$. The rate of change of N is just $\frac{dN}{dt}$. So, our differential equation would look something like

$$\frac{dN}{dt} = -\lambda N(t)$$

since we know the derivative will be negative. This is the principle behind carbon dating...if we can solve this equation, and we know how many C-14 atoms were present in a sample before it died and how many are there now (say N_0), we can solve $N(t) = N_0$ for t and know how old the sample is. \square

EXAMPLE 1.5 (Newton's Law of Cooling). In 1701, Newton published the following thermodynamic observation:

“The time rate of change of temperature in a body immersed in a constant temperature environment is proportional to the temperature difference between the body and the environment.”

Let's say $B(t)$ is the body's temperature. We'll refer to the constant external temperature as E . Newton's Law of Cooling, in mathematical form, becomes

$$\frac{dB}{dt} = \kappa(E - B(t))$$

where κ is some proportionality constant that will depend on the material of the body. \square

EXAMPLE 1.6 (Heat Equation). Suppose we have a (one-dimensional) rod of length l . We want to know how the temperature distribution on the rod evolves. We'll denote the temperature at position x and time t by $u(x, t)$. The heat equation (which we'll look at more closely towards the end of the semester) says

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

where k is some constant that indicates how quickly heat propagates through the material. What this says is that if we consider how the temperature is changing at a fixed point, it's proportional to the concavity of the temperature at that time around that point. \square

EXAMPLE 1.7 (Schrödinger Equation). This one's a doozy, so we won't worry about interpreting it, but I will mention it again later in the class:

$$i\hbar \frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Phi}{\partial x^2} + V(x)\Phi(x, t)$$

where Φ is the wave function, \hbar is Planck's constant divided by 2π , and $V(x)$ is a potential function. \square

4. Basic Definitions

Before we move on, we need to get some definitions out of the way so that we can talk about categories of equations in one shot.

DEFINITION 1.8. If a differential equation involves the derivatives of a function of a single variable, it is an *ordinary differential equation*. If it involves the partial derivatives of a function of several variables, it is a *partial differential equation*.

Newton's Law of Cooling and Radioactive Decay were ordinary differential equations, while the Heat Equation and Schrodinger's Equation were partial differential equations. If you haven't had multivariable calculus, don't worry: we'll talk more about partial derivatives when they become more relevant.

DEFINITION 1.9. The *order* of a differential equation is the highest derivative that occurs in the equation.

For example, both of the Newton's Law of Cooling and Radioactive Decay equations are first order. The Heat Equation and the Schrodinger Equation are second order.

$$y^{(4)} + 10y''' - 3y = 3t$$

is a fourth order ordinary differential equation.

DEFINITION 1.10. An ordinary differential equation is called *linear* if it has the form

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

for any functions $a_i(t), g(t)$. For a partial differential equation, substitute in all the relevant partial derivatives for the ordinary derivatives given above. A differential equation is *nonlinear* if it is not linear.

REMARK. Notice that linearity of the differential equation has nothing to do with the coefficient functions depending on t , only how the equation treats the unknown function y . All we do is take derivatives of y , multiply them by functions depending only on t , and add them together. If anything else is done, the equation is nonlinear. This is important to get clear from the get-go.

So, for example,

$$y'' + \frac{1}{\sin t}y' = 2t^2$$

is a linear equation, but

$$y'y^2 = 2t$$

is nonlinear.

5. General Strategy

Here are the questions to constantly keep in mind as we discuss various types of differential equations and the corresponding initial value problems:

(1) **Does a solution exist?**

Not all initial value problems have solutions. So we'd like to know ahead of time, without having to waste hours banging our heads against the wall, if there is a solution or not.

(2) **If a solution exists, how many are there?**

It's possible that there are several, or even infinitely, many solutions. This is usually a bad thing, since if we subject five identical rods to the same conditions, they should all have the same characteristics, such as temperature distributions. But if our initial value problem doesn't have a unique solution, this won't necessarily be the case, which doesn't make any physical sense. So we'd like to know when a unique solution exists.

(3) **How can we find any solution, if they exist?**

This is the question we will spend more of our time on, though we will discuss the first two questions in certain cases.