LECTURE 22

Higher Order Linear Equations

We've been talking about second order linear equations for a while. It turns out that there's nothing special about second order linear equations: all the techniques we discussed can be generalized to nth order linear differential equations. The main advantage to second order equations, as opposed to higher order equations, is that it's very easy to find the roots of quadratic polynomials, which isn't true for higher order equations.

1. Basic Concepts

Let's start with some basic concepts. The general form of an nth order linear equation is

$$P_n(t)y^{(n)} + P_{n-1}(t)y^{(n-1)} + \ldots + P_1(t)y' + P_0(t)y = G(t).$$

A lot of the facts here will require the leading coefficient function to be 1, so we'll assume our differential equation has the form

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = g(t). \tag{22.1}$$

We will also need some initial conditions. Since our equation is nth order, we need n of them:

$$y(t_0) = \overline{y}_0, \quad y'(t_0) = \overline{y}_1, \quad \ldots, \quad y^{(n-1)}(t_0) = \overline{y}_{n-1}. \tag{22.2}$$

Taken together, (22.1) and (22.2) form an nth order initial value problem.

1.1. Existence/Uniqueness. We have the following version of the Existence and Uniqueness Theorem:

THEOREM 22.1. *Suppose the functions $p_0(t), p_1(t), \ldots, p_{n-1}(t)$ and $g(t)$ in (22.1) are continuous on some interval (a, b) containing t_0. Then there is a unique solution to the initial value problem given by (22.1) and (22.2) which is defined for all t in (a, b).*

Notice that the statement of this theorem is a natural analogy to the first and second order linear versions that we've seen earlier: the only difference is the number of coefficient functions that need to remain continuous.

1.2. Homogeneous Equations. The general nth degree homogeneous equation has the form

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = 0. \tag{22.3}$$

Suppose we know that $y_1(t), y_2(t), \ldots, y_n(t)$ are all solutions to Equation 22.3. The Principle of Superposition says that then

$$y(t) = c_1y_1(t) + c_2y_2(t) + \ldots + c_ny_n(t)$$

is also a solution to (22.3) for any constants $c_1, \ldots, c_n$. The next question to ask is when this is a general solution for (22.3).

Recall from our discussion of second order equations that this linear combination is a general solution when, for any choice of t_0 in an appropriate interval (a, b) as in Theorem 22.1 and any choice of initial values $\overline{y}_0, \overline{y}_1, \ldots, \overline{y}_{n-1}$, we can find constants $c_1, c_2, \ldots, c_n$ so that the solution $y(t)$ to the initial value problem (22.1) and (22.2) can be written as

$$y(t) = c_1y_1(t) + c_2y_2(t) + \ldots + c_ny_n(t).$$

This means that we should be able to find constants $c_1, c_2, \dots, c_n$ that solve the following system of equations.

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) + \ldots + c_n y_n(t_0) &= \overline{y}_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) + \ldots + c_n y_n'(t_0) &= \overline{y}_1 \\ &\vdots \\ c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)}(t_0) + \ldots + c_n y_n^{(n-1)}(t_0) &= \overline{y}_{n-1} \end{aligned}$$

The most natural way (given the tedium of explicitly solving a system of n equations with n unknowns) is to use Cramer's Rule from linear algebra to obtain the solution. Upon doing so, we see that each of the constants has the same determinant as its denominator:

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

This determinant is called the *Wronskian* $W(y_1, y_2, \dots, y_n)$ of the functions $y_1, y_2, \dots, y_n$, again in direct analogy to the second order case. As the Wronskian is the denominator of the solutions c_i, we will have a solution to our system of equations as long as $W(y_1, y_2, \dots, y_n)(t)$ is nonzero at any value of $t = t_0$ that we choose to evaluate the Wronskian at. The following theorem summarizes our previous discussion.

THEOREM 22.2. *Suppose the functions $p_0(t), p_1(t), \dots, p_{n-1}(t)$, and $g(t)$ are all continuous on an interval (a, b) and that $y_1(t), y_2(t), \dots, y_n(t)$ are all solutions to eqrefnthorderhomog. If $W(y_1, y_2, \dots, y_n)(t) \neq 0$ for every t in (a, b), then $y_1(t), y_2(t), \dots, y_n(t)$ form a* fundamental set of solutions *and the general solution to* (22.3) *is*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \ldots + c_n y_n(t).$$

Recall as well that if a set of solutions forms a fundamental set of solutions, then they must be *linearly independent.* n functions $y_1(t), y_2(t), \dots, y_n(t)$ are linearly independent if the only constants $c_1, c_2, \dots, c_n$ satisfying

$$c_1 y_1(t) + c_2 y_2(t) + \ldots + c_n y_n(t) = 0$$

are $c_1 = c_2 = \ldots = c_n = 0$. As before, however, this doesn't go both ways: a set of linearly independent solutions are not necessarily a fundamental set of solutions.

1.3. Nonhomogeneous Equations. Let's briefly discuss how to find solutions to the higher order nonhomogeneous equation (22.1). Once again, all we're doing is stating the nth order versions of results that we've seen in the 2nd order case.

THEOREM 22.3. *Suppose that $Y_1(t)$ and $Y_2(t)$ are two solutions to* (22.1) *and that $y_1(t), y_2(t), \dots, y_n(t)$ are a fundamental set of solutions to the* associated homogeneous equation (22.3). *Then*

$$Y_1(t) - Y_2(t)$$

is a solution to (22.3) *and can be written in the form*

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) + \ldots + c_n y_n(t).$$

Now, if $Y(t)$ is the general solution to (22.1) and $Y_p(t)$ is any solution to (22.1), then we must have

$$Y(t) = c_1 y_1(t) + c_2 y_2(t) + \ldots + c_n y_n(t) + Y_p(t) = y_c(t) + Y_p(t),$$

where $y_c(t)$ is called the *complimentary solution* and $Y_p(t)$ is called a *particular solution.*

2. Homogeneous Equations

Next, we'll discuss the differences between finding solutions for nth degree equations and 2nd degree equations. What we'll see is that the methods involved are the same, but there's a bit more work involved.

We're going to restrict our attention to the constant coefficient case, since solving nonconstant coefficient equations is rather difficult. In the second order case, the only technique we know is Reduction of Order, which lets us use a known solution to reduce a second order equation to a first order linear equation, which we can always solve (ignoring whether we can compute all the necessary integrals). For this technique to be workable for an nth order equation, however, we would need to know $n-1$ solutions, which is quite a difficult task.

So, we're considering the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0. \tag{22.4}$$

As in the second order case, we might guess that solutions to this equation have the form $y(t) = e^{rt}$. Plugging in to the equation yields

$$e^{rt}\left(a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0\right) = 0$$

or

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0. \tag{22.5}$$

(22.5) is called the *characteristic equation* of (22.4) and we've just seen that if r is a solution of the characteristic equation, then $y(t) = e^{rt}$ is a solution to (22.4). Now, we know that, counting multiplicities, any nth degree polynomial, such as (22.5), has n roots (this is the statement of the Fundamental Theorem of Algebra). These roots may be real or complex, and they may have varying multiplicities.

This is a big difference from the second order case. Because the characteristic equation for a second order differential equation was a quadratic, with two roots (counting multiplicities), we had a very limited number of potential combinations: we could have two distinct real roots r_1 and r_2, a pair of complex roots that came in conjugate pairs $\alpha \pm i\beta$, or a real root r which had muliplicity greater than one, and no combinations thereof. For the nth order case, however, while these are still the main cases we will need to consider, the roots may come in any number of combinations of these cases. A twelfth order differential equation, for example, might have a characteristic equation might have three distinct real roots, a pair of conjugate complex roots with multiplicity one, another conjugate pair of complex roots, this time having multiplicity two, and another real root with multiplicity three.

The goal is to form a fundamental set of n solutions. This is entirely analogous to the second order case; it can just get slightly more complicated as cases build up. Let's start by supposing we have k real roots $r_1, r_2, \ldots, r_k$, all of which only have multiplicity one. The solutions corresponding to each of these roots are

$$e^{r_1 t}, \qquad e^{r_2 t}, \qquad \ldots, \qquad e^{r_k t}.$$

It's not hard (though it is a big tedious) to verify that these are certainly linearly independent; just compute their Wronskian.

Now suppose that we have a root r which has multiplicity k. We want to generate k linearly independent solutions from this one root. The way we do this is analogous to what we did for a repeated root in the second order case, namely, we have the following k solutions.

$$e^{rt}, \qquad te^{rt}, \qquad \ldots, \qquad t^{k-1}e^{rt}$$

Again, showing that these are linearly independent is straightforward.

That takes care of any real roots our characteristic equation might have. What about complex roots? Suppose we have a pair of conjugate complex roots $\alpha \pm i\beta$ that only have multiplicity one. In this case, we get the standard pair of solutions

$$e^{\alpha t}\cos(\beta t), \qquad e^{\alpha t}\sin(\beta t).$$

But what happens if we have a pair of complex roots $\alpha \pm i\beta$ which have multiplicity k (which can happen if our equation is fourth order or higher)? The same work as in the real repeated roots case gives the following set of $2k$ complex-valued solutions.

$$\begin{array}{llll} e^{(\alpha+i\beta)t}, & te^{(\alpha+i\beta)t}, & \dots, & t^{k-1}e^{(\alpha+i\beta)t} \\ e^{(\alpha-i\beta)t}, & te^{(\alpha-i\beta)t}, & \dots, & t^{k-1}e^{(\alpha+i\beta)t} \end{array}$$

Of course, we want real-valued solutions, not complex-valued ones. In the case where $k = 1$, we used Euler's formula and some algebra to convert the first complex-valued solution into a pair of real-valued solutions. Here, we'll do this on the first list of complex-valued solutions to get the following list of $2k$ real-valued solutions:

$$e^{\alpha t}\cos(\beta t), \quad e^{\alpha t}\sin(\beta t), \quad te^{\alpha t}\cos(\beta t), \quad te^{\alpha t}\sin(\beta t), \quad \dots, \quad t^{k-1}e^{\alpha t}\cos(\beta t), \quad t^{k-1}e^{\alpha t}\cos(\beta t).$$

Once again, verification of linear independence is not hard, but a bit tedious.

In the end, we will have n linearly independent solutions, as desired. So we can then form a general solution. There is, however, one additional complication that is associated with higher order equations. In the second order case, the characteristic equations were always quadratic. This meant that finding the roots was never a problem: if we couldn't factor the equation, we could always complete the square or use the quadratic formula.

Now, however, our characteristic equations will have degrees 3 or greater. Finding roots for these by hand is quite difficult (there are cubic and quartic formulae, but they're very unwieldy, and it's a very deep mathematical fact that there are no general formulae for roots of polynomials with degree ≥ 5). In fact, if the roots aren't rational, it can be almost impossible to find all of them by hand. There are some tricks for finding rational roots, but they generally involve guesswork and plugging in several possibilities to see which work.

Let's do some examples to see how all of this comes together.

Example 22.1. *Solve the following initial value problem.*

$$y''' + 3y'' - 4y' - 12y = 0 \qquad y(0) = 1, y'(0) = 3, y''(0) = -2$$

The characteristic equation is

$$r^3 + 3r^2 - 4r - 12 = 0$$
$$(r-2)(r+2)(r+3) = 0.$$

So there are three distinct real roots $r_1 = 2$, $r_2 = -2$, and $r_3 = -3$. The general solution is then

$$y(t) = c_1e^{2t} + c_2e^{-2t} + c_3e^{-3t}.$$

If we plug in our initial conditions, we get the following system of equations.

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + c_3 \\ 3 &= y'(0) = 2c_1 - 2c_2 - 3c_3 \\ -2 &= y''(0) = 4c_1 + 4c_2 + 9c_3 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} c_1 &= \frac{19}{20} \\ c_2 &= \frac{5}{4} \\ c_3 &= -\frac{6}{5} \end{aligned}$$

The particular solution is

$$y(t) = \frac{19}{20}e^{2t} + \frac{5}{4}e^{-2t} - \frac{6}{5}e^{-3t}.$$

□

EXAMPLE 22.2. *Find the general solution to the following differential equation.*

$$y^{(5)} - y^{(4)} - 5y''' + y'' + 8y' + 4y = 0.$$

The characteristic equation is

$$r^5 - r^4 - 5r^3 + r^2 + 8r + 4 = 0$$
$$(r-2)^2(r+1)^3 = 0.$$

Here we have one root, $r_1 = 2$, with multiplicity 2, and a second, $r_2 = -1$, with multiplicity 3. The first root will give us two solutions and the second three, so we'll end up with the desired five solutions by multiplying by t where appropriate.

So the general solution is

$$y(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 e^{-t} + c_4 t e^{-t} + c_5 t^2 e^{-t}.$$

□

EXAMPLE 22.3. *Find the general solution to the following differential equation.*

$$y^{(7)} + 49y^{(5)} - 104y^{(4)} + 900y''' - 1200y'' + 4000y' = 0.$$

The characteristic equation is

$$r^7 + 49r^5 - 104r^4 + 900r^3 - 1200r^2 + 4000r = 0$$
$$r\left(r^2 + 4r + 40\right)\left(r^2 - 2r + 10\right)^2 = 0.$$

So we have a real root $r = 0$ with multiplicity 1, a pair of complex roots $r = -2 \pm 6i$ with multiplicity 1, and a pair of complex roots $r = 1 \pm 3i$ with multiplicity 2. Our general solution is then

$$y(t) = c_1 + c_2 e^{-2t}\cos(6t) + c_3 e^{-2t}\sin(6t) + c_4 e^t\cos(3t) + c_5 e^t\sin(3t) + c_6 t e^t\cos(3t) + c_7 t e^t\sin(3t).$$

□