LECTURE 45

Heat Equation Problems

1. Review

In Lecture 40, we saw that product solutions to the heat equation with homogeneous Dirichlet boundary conditions problem

$$\begin{gathered} u_t = k u_{xx} \qquad 0 < x < l \quad t > 0 \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = f(x) \end{gathered} \tag{45.1}$$

had the form

$$u_n(x,t) = B_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right) \qquad n = 1,2,3,\ldots \tag{45.2}$$

Taking linear combinations of these (over each n) gives a general solution to (45.1)

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right)}. \tag{45.3}$$

Setting $t = 0$, this implies that we must have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right);$$

in other words, the coefficients in the general solution (45.3) for the given initial condition are the Fourier *sine* coefficients of $f(x)$ on $(0,l)$, which are given by

$$\boxed{B_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)\, dx.} \tag{45.4}$$

We also saw that if instead we have a problem with homogeneous Neumann boundary conditions

$$\begin{gathered} u_t = k u_{xx} \qquad 0 < x < l, \quad t > 0 \\ u_x(0,t) = u_x(l,t) = 0 \\ u(0,t) = f(x) \end{gathered}$$

, the product solutions had the form

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \cos\left(\frac{n\pi x}{l}\right) \qquad n = 0,1,2,3,\ldots \tag{45.5}$$

and the general solution to (1) has the form

$$\boxed{u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \cos\left(\frac{n\pi x}{l}\right).} \tag{45.6}$$

With $t = 0$, this means that the initial condition must satisfy

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right),$$

and so the coefficients in (45.6) for a particular initial condition are the Fourier *cosine* coefficients of $f(x)$, given by

$$\boxed{A_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)\,dx.} \tag{45.7}$$

One way of thinking about this difference is that, given the initial data $u(x,0) = f(x)$, the Dirichlet conditions in (45.1) specify the *odd extension* of $f(x)$ as the desired periodic solution, while the Neumann conditions in (1) specify the *even extension*. This should make some sense: as we've discussed, odd functions must have $f(0) = 0$, while even functions must have $f'(0) = 0$.

So, to solve a homogeneous heat equation problem, we begin by identifying the type of boundary conditions we have. If we have Dirichlet conditions, we know our solution will have the form (45.3), and all we have to do is compute the Fourier sine coefficients of $f(x)$ (45.4). Similarly, if we have Neumann conditions, we know the solution has the form (45.6) and we have to compute the Fourier cosine coefficients of $f(x)$ (45.7).

Remark. Observe that for any homogeneous Dirichlet problem (45.1), the temperature distribution (45.3) will go to 0 as $t \to \infty$. This should make sense: these boundary conditions have a physical interpretation where we keep the ends of our rod at freezing without regulating the heat flow in or out of the endpoints. As a result, if the interior of the rod is initially above freezing, that heat will radiate towards the endpoints and into our reservoirs at the endpoints. On the other hand, if the interior of the rod is below freezing, heat will come from the reservoirs at the endpoints and warm it up until the temperature is all uniform.

For the Neumann problem (1), the temperature distribution (45.6) will converge to $\frac{1}{2}A_0$. Again, this should make some physical sense: these boundary conditions correspond to a situation where we have insulated ends, as we're preventing any heat from leaving the bar. Thus all the heat energy will move around inside the rod until the temperature is uniform.

2. Examples

Example 45.1. *Solve the initial value problem*

$$\begin{aligned} u_t &= 3u_{xx} \qquad 0 < x < 2 \quad t > 0 \\ u(0,t) &= u(2,t) = 0 \\ u(x,0) &= 20. \end{aligned}$$

This problem has homogeneous Dirichlet conditions, so our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-3\left(\frac{n\pi}{2}\right)^2 t}\sin\left(\frac{n\pi x}{2}\right).$$

The coefficients for the particular solution are the Fourier sine coefficients of $u(x,0) = 20$, so we have

$$\begin{aligned} B_n &= \frac{2}{2}\int_0^2 20\sin\left(\frac{n\pi x}{2}\right)\,dx \\ &= \left[-\frac{40}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 \\ &= -\frac{40}{n\pi}\left(\cos(n\pi) - \cos(0)\right) \\ &= \frac{40}{n\pi}\left(1 + (-1)^{n+1}\right) \end{aligned}$$

and the solution to the problem is

$$u(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n}e^{-\frac{3n^2\pi^2}{4}t}\sin\left(\frac{n\pi x}{2}\right).$$

□

EXAMPLE 45.2. *Solve the initial value problem*

$$\begin{gathered} u_t = 3u_{xx} \qquad 0 < x < 2, \quad t > 0 \\ u_x(0,t) = u_x(2,t) = 0 \\ u(x,0) = 3x. \end{gathered}$$

This problem has homogeneous Neumann conditions, so by (45.6) our general solution is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-3\left(\frac{n\pi}{2}\right)^2 t}\cos\left(\frac{n\pi x}{2}\right).$$

The coefficients for the particular solution are the Fourier sine coefficients of $u(x,0) = 3x$, so we have

$$\begin{aligned} A_0 &= \frac{2}{2}\int_0^2 3x\,dx = 6 \\ A_n &= \frac{2}{2}\int_0^2 3x\cos\left(\frac{n\pi x}{2}\right)\,dx \\ &= \left[-\frac{6x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{12}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 \\ &= -\frac{12}{n\pi}\cos(n\pi) \\ &= \frac{12}{n\pi}(-1)^{n+1} \end{aligned}$$

and the solution to the problem is

$$u(x,t) = \frac{3}{2} + \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{3n^2\pi^2}{4}t}\cos\left(\frac{n\pi x}{2}\right).$$

□

EXAMPLE 45.3. *Solve the initial value problem*

$$\begin{gathered} u_t = 4u_x x \qquad 0 < x < 2\pi, \quad t > 0 \\ u(0,t) = u(2\pi,t) = 0 \\ u(x,0) = \begin{cases} 1 & 0 < x \le \pi \\ x & \pi < x < 2\pi \end{cases} \end{gathered}$$

We have Dirichlet conditions, so by (45.3) the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t}\sin\left(\frac{nx}{2}\right),$$

and the coefficients are given by

$$\begin{aligned}
B_n &= \frac{2}{2\pi}\left(\int_0^{\pi} \sin\left(\frac{nx}{2}\right)\,dx + \int_{\pi}^{2\pi} x\sin\left(\frac{nx}{2}\right)\,dx\right)\\
&= -\frac{2}{n\pi}\cos\left(\frac{nx}{2}\right)\Big|_0^{\pi} - \frac{2x}{n\pi}\cos\left(\frac{nx}{2}\right)\Big|_{\pi}^{2\pi} + \frac{4}{n^2\pi}\sin\left(\frac{nx}{2}\right)\Big|_{\pi}^{2\pi}\\
&= -\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(0)\right) - \frac{4}{n}\cos(n\pi) + \frac{2}{n}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)\\
&= -\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) + \frac{4}{n}(-1)^{n+1} + \frac{2}{n}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)\\
&= \frac{2}{n}\left(-\frac{1}{\pi}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) + 2(-1)^{n+1}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)
\end{aligned}$$

and the solution is given by

$$u(x,t) = 2\sum_{n=1}^{\infty}\frac{1}{n}\left(-\frac{1}{\pi}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) + 2(-1)^{n+1}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)e^{-n^2t}\sin\left(\frac{nx}{2}\right)$$

□