LECTURE 13

Fundamental Sets of Solutions

Over the past few lectures, our focus has been on constructing general solutions to certain linear differential equations. To do this, we've needed to find two "different" solutions $y_1(t)$ and $y_2(t)$ so that their linear combination is a general solution. But how do we know when y_1 and y_2 are "different" enough for this to be the case? The time has come to confront this, rather than just saying solutions are sufficiently "different." What is the precise condition on y_1 and y_2 that makes them "different" enough to form a general solution?

1. Existence and Uniqueness

The first question we should ask is if, given an initial value problem involving a linear second order equation, a solution exists. We've already commented that the answer is yes (when discussing the first order case), but let's just repeat:

THEOREM 13.1. *Consider the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t) \qquad y(t_0) = y_0, y'(t_0) = y_0'.$$

If $p(t)$, $q(t)$, and $g(t)$ are all continuous on some interval (a, b) such that $a < t_0 < b$, then the initial value problem has a unique solution defined on (a, b).

2. The Wronskian

Let's suppose we're working with the following initial value problem

$$p(t)y'' + q(t)y' + r(t)y = 0 \tag{13.1a}$$

$$y(t_0) = y_0 \qquad y'(t_0) = y_0' \tag{13.1b}$$

and that we know two solutions $y_1(t)$ and $y_2(t)$. Since the differential equation (13.1a) is linear and homogeneous, the Principle of Superposition says that any linear combination

$$y(t) = c_1y_1(t) + c_2y_2(t) \tag{13.2}$$

is also a solution. We want to know when this is a general solution. For this to be the case, it must satisfy the general initial conditions (13.1b). In other words, without any restrictions (beyond requiring t_0 to not be a point of discontinuty for the coefficient functions, so that Theorem 13.1 applies) we should be able to find constants c_1 and c_2 that work for the given initial conditions.

We start by differentiating Equation 13.2 and plugging in the initial conditions.

$$\begin{aligned} y_0 &= y(t_0) = c_1y_1(t_0) + c_2y_2(t_0) \\ y_0' &= y'(t_0) = c_1y_1'(t_0) + c_2y_2'(t_0) \end{aligned} \tag{13.3}$$

Let's solve this system of equations. We have that

$$c_1 = \frac{y_0 - c_2y_2(t_0)}{y_1(t_0)}.$$

Thus:

$$\begin{aligned} y_0' &= \frac{y_0 y_1'(t_0) - c_2 y_2(t_0) y_1'(t_0)}{y_1(t_0)} + c_2 y_2'(t_0) \\ &= \frac{y_0 y_1'(t_0) - c_2 y_2(t_0) y_1'(t_0) + c_2 y_2'(t_0) y_1(t_0)}{y_1(t_0)} \end{aligned}$$

and we compute

$$\begin{aligned} c_2 &= \frac{y_0' y_1(t_0) - y_0 y_1'(t_0)}{y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0)} \\ c_1 &= \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0)}. \end{aligned}$$

Notice that c_1 and c_2 have the same quantity in their denominators: $y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$. So the only time we won't be able to solve for c_1 and c_2 is when this quantity is zero.

Definition 13.1. The quantity

$$W(y_1, y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0)$$

is called the *Wronskian*[1] *of* y_1 *and* y_2 *at* t_0.

Remark.

(1) When it's clear what the two functions in question are, we will often just denote the Wronskian by W.

(2) The notation we've used for the Wronskian seems to indicate that we can think of $W(y_1, y_2)(t)$ as a function dependant on t. This is definitely the case. The "Wronskian of y_1 and y_2", $W(y_1, y_2)(t)$, is a function of t and can be evaluated at any t where y_1 and y_2 are defined. This is important, since for the two solutions y_1 and y_2 to satisfy the general initial conditions (13.1b), we'll need $W(y_1, y_2)$ to be nonzero at any value of t_0 where Theorem 13.1 applies.

(3) We could have also solved the system of equations (13.3) by using Cramer's Rule from linear algebra.

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2(t_0) \end{vmatrix}}$$

Here

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

is the determinant of the 2x2 matrix. It's alright if you haven't taken linear algebra; the formula is easy enough. Notice that the denominators of c_1 and c_2 here are exactly the same as before. So we also have

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}.$$

[1] The Wronskian is named after the Polish philosopher and mathematician Józef Hoëne-Wroński (1778-1853). Wroński focused on applying philosophy to mathematics. The Wronskian has its origin in Wroński's attempt to supplant the use of infinite series representations of functions with his own ideas. For many years, Wroński's ideas were given short shrift, primarily because his contemporaries felt that he had too high an opinion of himself and his research and his violent reaction to criticism. It's now felt that while much of his work was wrong, there were flashes of brilliant insight in his papers. Shortly before his death, he reportedly said "God Almighty, there's still so much more I wanted to say!"

We will generally express the Wronskian in this determinant form.

Let's recall what the situation is. We wanted to know when two solutions to (13.1a) were different enough to form a general solution. The two solutions will form a general solution if they satisfy the general initial conditions (13.1b). The above computation showed that this will be the case so long as

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \neq 0.$$

If $y_1(t)$ and $y_2(t)$ are solutions to (13.1a) and $W(y_1, y_2)(t) \neq 0$, then the two solutions are said to be a *fundamental set of solutions* for (13.1a) and the general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

In other words, two solutions are "different" enough to form a general solution if they are a fundamental set of solutions. Let's check some of the claims we made earlier.

Example 13.2. *If r_1 and r_2 are distinct real roots of the characteristic equation for $ay'' + by' + cy = 0$, check that*

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t}$$

form a fundamental set of solutions.

To show this, we'll need to compute the Wronskian and see that it isn't zero.

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = r_2 e^{(r_2 + r_1)t} - r_1 e^{(r_2 + r_1)t} = e^{(r_2 - r_1)t}(r_2 - r_1)$$

Since exponentials are never zero and $r_2 \neq r_1$, we conclude that $W \neq 0$ and so, as claimed earlier, y_1 and y_2 form a fundamental set of solutions for the differential equation and the general solution is in fact

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

□

Example 13.3. The first example we did during the Reduction of Order lecture was to find a second solution to

$$2t^2 y'' + ty' - 3y = 0$$

given that $y_1(t) = t^{-1}$ is a solution. Let's show that $y_1(t)$ and $y_2(t) = t^{\frac{3}{2}}$ form a fundamental set of solutions. To do this, we compute the Wronskian.

$$W = \begin{vmatrix} t^{-1} & t^{\frac{3}{2}} \\ -t^{-2} & \frac{3}{2} t^{\frac{1}{2}} \end{vmatrix} = \frac{3}{2} t^{-\frac{1}{2}} + t^{-\frac{1}{2}} = \frac{5}{2\sqrt{t}}$$

Thus $W \neq 0$, so they are a fundamental set of solutions.

Notice that we can't plug $t = 0$ into the Wronskian. This is alright, since we can't plug $t = 0$ into the solutions, either.

Thus the general solution is, in fact,

$$y(t) = c_1 t^{-1} + c_2 t^{\frac{3}{2}}.$$

□

Example 13.4. The other reduction of order example we did involved the differential equation

$$t^2 y'' + 2ty' - 2y = 0.$$

We were given that $y_1(t) = t$ was a solution and found that $y_2(t) = t^{-2}$ was another solution. Again, let's check that these two solutions in fact form a fundamental set of solutions.

$$W = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0.$$

So the solutions are a fundamental set of solutions, and the general solution is

$$y(t) = c_1 t + c_2 t^{-2}.$$

□

The final question we need to ask here is how we know if a fundamental set of solutions will exist for a given differential equations. The following theorem gives the answer.

THEOREM 13.2. *Consider the differential equation*

$$y'' + p(t)y' + q(t) = 0$$

where $p(t)$ and $q(t)$ are continuous on some interval (a, b). Suppose $a < t_0 < b$. If $y_1(t)$ is a solution satisfying the initial conditions

$$y(t_0) = 1 \qquad y'(t_0) = 0$$

and $y_2(t)$ is a solution satisfying

$$y(t_0) = 0 \qquad y'(t_0) = 1,$$

then $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions.

In general, we won't want to use this theorem to compute the fundamental set of solutions, since the particular solutions we get from those initial conditions might not be as nice as would like. The main importance it has is that it assures us that, as long as the coefficient functions $p(t)$ and $q(t)$ are continuous, a fundamental set of solutions will exist.

3. Linear Independence

The motivation for introducing the Wronskian was to study when a general linear combination of two solutions to the differential equation (13.1a) was the general solution. We found that if this is the case, the Wronskian will be nonzero. But it's not yet clear why we should expect the Wronskian of two functions to be either zero or nonzero.

To do this, we need to introduce a couple of new notions. Suppose we have two functions $f(t)$ and $g(t)$. We can always consider the equation

$$c_1 f(t) + c_2 g(t) = 0. \tag{13.4}$$

Notice that $c_1 = 0$ and $c_2 = 0$ always solve this equation, regardless of what f and g are.

DEFINITION 13.5. If there are nonzero constants c_1 and c_2 such that Equation 13.4 is satisfied for all t, then the functions f and g are said to be *linearly dependant.* On the other hand, if the only constants for which Equation 13.4 is true are $c = 0$ and $k = 0$, then f and g are said to be *linearly independent.*

REMARK. Two functions are linearly dependant precisely when they are constant multiples of each other. If Equation 13.4 is satisfied by nonzero c_1 and c_2, we have

$$f(t) = -\frac{c_2}{c_1} g(t).$$

This is often useful in deciding whether or not two functions are linearly dependant or not.

Example 13.6. *Determine if the following pairs of functions are linearly dependant or independant.*

(1) $f(x) = 9\cos(2x) \qquad g(x) = 2\cos^2(x) - 2\sin^2(x)$
(2) $f(t) = 2t^2 \qquad g(t) = t^4$

(1) Let's start by writing down Equation 13.4 for these functions. We have

$$9c_1\cos(2x) + 2c_2(\cos^2(x) - \sin^2(x)).$$

We're looking for nonzero constants c_1 and c_2 that make this true. In this case, we have a trig identity: $\cos(2x) = \cos^2(x) - \sin^2(x)$. Then our equation becomes

$$\begin{aligned} 9c_1\cos(2x) + 2c_2\cos(2x) &= 0 \\ (9c_1 + 2c_2)\cos(2x) &= 0. \end{aligned}$$

This equation is true for any c_1 and c_2 satisfying $9c_1 + 2c_2 = 0$. So we can take, for example, $c_1 = 2$ and $c_2 = -9$, though there are infinitely many different pairs that will work. We conclude that the two functions are linearly dependant.

(2) Once again, we start by writing down Equation 13.4.

$$2c_1t^2 + c_2t^4 = 0$$

There's no nice identity or formula we can use, so let's start by observing that if this is true, we can differentiate both sides and also get a true equation. This gives the system of equations

$$\begin{aligned} 2c_1t^2 + c_2t^4 &= 0 \\ 4c_1t + 4c_2t^3 &= 0. \end{aligned}$$

We will solve this system for c_1 and c_2. The second equation tells us that $c_1 = -c_2t^2$. Plugging this into the first equation gives

$$\begin{aligned} 2(-c_2t^2)t^2 + c_2t^4 &= 0 \\ -c_2t^4 &= 0. \end{aligned}$$

This is only true for all t if $c_2 = 0$ which in turn tells us that $c_1 = 0$. Hence the functions are linearly independent.

□

As you can see, this can be a fairly involved process. It's also not always clear how to proceed. It would be nice to have a criterion for linear independence that might let us avoid these computations. This is where the Wronskian helps.

Theorem 13.3. *Given two functions $f(t)$ and $g(t)$ which are differentiable on some interval (a, b),*

(1) *If $W(f, g)(t_0) \neq 0$ for some $a < t_0 < b$, then $f(t)$ and $g(t)$ are linearly independent on (a, b) and*
(2) *If If $f(t)$ and $g(t)$ are linearly dependent on (a, b), then $W(f, g)(t) = 0$ for all $a < t < b$.*

Remark. Be careful: the preceding theorem *does not say* that if $W(f, g)(x) = 0$ then f and g are linearly dependent. It's possible for two linearly independent functions to have a zero Wronskian.

Let's use this theorem to double check our earlier example.

Example 13.7.

(1) Let's consider $f(x) = 9\cos(2x)$ and $g(x) = 2\cos^2(x) - 2\sin^2(x)$.

$$\begin{aligned} W &= \begin{vmatrix} 9\cos(2x) & 2\cos^2(x) - 2\sin^2(x) \\ -18\sin(2x) & -4\cos(x)\sin(x) - 4\cos(x)\sin(x) \end{vmatrix} \\ &= \begin{vmatrix} 9\cos(2x) & 2\cos(2x) \\ -18\sin(2x) & -4\sin(2x) \end{vmatrix} \\ &= -36\cos(2x)\sin(2x) + 36\cos(2x)\sin(2x) &= 0 \end{aligned}$$

So we got zero, as we should have, since we know that these two functions are linearly dependant. Note that we still had to make heavy use of trig formulae.

(2) Now let's take $f(t) = 2t^2$ and $g(t) = t^4$.

$$\begin{aligned} W &= \begin{vmatrix} 2t^2 & t^4 \\ 4t & 4t^3 \end{vmatrix} \\ &= 8t^5 - 4t^5 \\ &= 4t^5 \end{aligned}$$

The Wronskian will be nonzero so long as $t = 0$, which is ok. We just don't want it to be zero for all t. □