LECTURE 44

More Fourier Series Convergence

1. Fourier Convergence (Again)

1.1. Recap. Last lecture, we discussed convergence issues for the Fourier sine equation

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$$

on the interval $(0, l)$. We saw that this sine series is identical to the full Fourier series of the odd extension of $f(x)$, $f_{\text{odd}}(x)$, on $(-l, l)$, and so will converge to the periodic extension of $f_{\text{odd}}(x)$ as given by the Fourier Convergence Theorem.

1.2. Fourier Cosine Series. Now, let's consider what happens for the Fourier cosine series of $f(x)$ on $(0, l)$. This is directly analogous to the sine series case. Every term in the cosine series has the form

$$A_n \cos\left(\frac{n\pi x}{l}\right)$$

and hence is even, so the entire cosine series is even. So the cosine series must converge on $(-l, l)$ to an even function which coincides on $(0, l)$ with $f(x)$. This must be the even extension,

$$f_{\text{even}}(x) = \begin{cases} f(x) & 0 < x < l \\ f(-x) & -l < x < 0. \end{cases} \tag{44.1}$$

Notice that this definition does not specify the value of the function at zero; the only restriction on an even function at zero is that, if it exists, the derivative should be zero.

It's straightforward enough to show that the Fourier coefficients of $f_{\text{even}}(x)$ coincide with the Fourier cosine coefficients of $f(x)$. The Euler-Fourier formulas give:

$$\begin{aligned} A_n &= \frac{1}{l}\int_{-l}^{l} f_{\text{even}}(x)\cos\left(\frac{n\pi x}{l}\right)\,dx \\ &= \frac{2}{l}\int_{0}^{l} f_{\text{even}}(x)\cos\left(\frac{n\pi x}{l}\right)\,dx && \text{since } f_{\text{even}}(x)\cos\left(\frac{n\pi x}{l}\right) \text{ is even} \\ &= \frac{2}{l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)\,dx \end{aligned}$$

which are the Fourier cosine coefficients of $f(x)$ on $(0, l)$;

$$\begin{aligned} B_n &= \frac{1}{l}\int_{-l}^{l} f_{\text{even}}(x)\sin\left(\frac{n\pi x}{l}\right)\,dx \\ &= 0 && \text{since } f_{\text{even}}(x)\sin\left(\frac{n\pi x}{l}\right) \text{ is odd.} \end{aligned}$$

Thus the Fourier cosine series of $f(x)$ on $(0, l)$ can be considered as the Fourier expansion of $f_{\text{even}}(x)$ on $(-l, l)$, and therefore also as an expansion of the periodic extension of $f_{\text{even}}(x)$. It will converge as in the Fourier Convergence Theorem to this periodic extension.

This also means that if we want to compute the Fourier series of an even function, we can just compute the Fourier cosine series of its restriction to $(0, l)$. It's very important, however, that this only be attempted if the function we're starting with is even.

EXAMPLE 44.1. *Write down the even extension of $f(x) = l - x$ on $(0, l)$ and compute its Fourier series.*

The even extension, by (44.1), will be

$$f_{\text{even}}(x) = \begin{cases} l - x & 0 < x < l \\ l + x & -l < x < 0. \end{cases}$$

Its Fourier series is the same as the Fourier cosine series of $f(x)$ by the previous discussion, so we can just compute those coefficients.

Thus we have

$$f_{\text{even}}(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right),$$

where

$$\begin{aligned} A_0 &= \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l (l - x)\,dx = l \\ A_n &= \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)\,dx \\ &= \frac{2}{l}\int_0^l (l - x)\cos\left(\frac{n\pi x}{l}\right)\,dx \\ &= \frac{2}{l}\left[\frac{l(l-x)}{n\pi}\sin\left(\frac{n\pi x}{l}\right) - \frac{l^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l \\ &= \frac{2}{l}\left(\frac{l^2}{n^2\pi^2}(-\cos(n\pi) + \cos(0))\right) \\ &= \frac{2l}{n^2\pi^2}\left((-1)^{n+1} + 1\right). \end{aligned}$$

So we have

$$f_{\text{even}}(x) = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2}\left((-1)^{n+1} + 1\right).$$

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EXAMPLE 44.2. *Write down the even extension of*

$$f(x) = \begin{cases} \frac{3}{2} & 0 \le x < \frac{3}{2} \\ x - \frac{3}{2} & \frac{3}{2} \le x \le 3 \end{cases}$$

and compute its Fourier series.

Using (44.1), we see that the even extension is

$$f_{\text{even}}(x) = \begin{cases} x - \frac{3}{2} & \frac{3}{2} < x < 3 \\ \frac{3}{2} & 0 \le x < \frac{3}{2} \\ \frac{3}{2} & -\frac{3}{2} < x < 0 \\ -x - \frac{3}{2} & -3 \le x \le -\frac{3}{2}. \end{cases}$$

We just need to compute the Fourier cosine coefficients of the original $f(x)$ on $(0,3)$.

$$\begin{aligned}
A_0 &= \frac{2}{3}\int_0^3 f(x)\,dx \\
&= \frac{2}{3}\left(\int_0^{\frac{3}{2}} \frac{3}{2}\,dx + \int_{\frac{3}{2}}^3 x - \frac{3}{2}\,dx\right) \\
&= \frac{2}{3}\left(\frac{9}{4} + \frac{9}{8}\right) = \frac{9}{4} \\
A_n &= \frac{2}{3}\int_0^3 f(x)\cos\left(\frac{n\pi x}{3}\right)\,dx \\
&= \frac{2}{3}\left(\int_0^{\frac{3}{2}} \frac{3}{2}\cos\left(\frac{n\pi x}{3}\right)\,dx + \int_{\frac{3}{2}}^3 \left(x - \frac{3}{2}\right)\cos\left(\frac{n\pi x}{3}\right)\,dx\right) \\
&= \frac{2}{3}\left(\frac{9}{2n\pi}\sin\left(\frac{n\pi x}{3}\right)\Bigg|_0^{\frac{3}{2}} + \frac{3\left(x-\frac{3}{2}\right)}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\Bigg|_{\frac{3}{2}}^3 + \frac{9}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right)\Bigg|_{\frac{3}{2}}^3\right) \\
&= \frac{2}{3}\left(\frac{9}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{9}{n^2\pi^2}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right)\right) \\
&= \frac{6}{n\pi}\left(\frac{1}{2}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right)\right) \\
&= \frac{6}{n\pi}\left(\frac{1}{n\pi}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right) + \frac{1}{2}\sin\left(\frac{n\pi}{2}\right)\right).
\end{aligned}$$

So the Fourier series is

$$f_{\text{even}}(x) = \frac{9}{8} + \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n\pi}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right) + \frac{1}{2}\sin\left(\frac{n\pi}{2}\right)\right)\cos\left(\frac{n\pi x}{3}\right).$$

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