LECTURE 42

# 2 Fourier 2 Series

Last class, we derived the Euler-Fourier formulas for the coefficients of the Fourier series of a function $f(x)$. For the *Fourier sine series* of $f(x)$ on the interval $(0, l)$,

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right), \tag{42.1}$$

we have

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)\,dx, \tag{42.2}$$

where $n = 1, 2, 3, \ldots$ For the *Fourier cosine series* of $f(x)$ on $(0, l)$,

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right), \tag{42.3}$$

the coefficients are given by

$$A_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)\,dx, \tag{42.4}$$

$n = 0, 1, 2, \ldots$ Finally, for the *(full) Fourier series* of $f(x)$, which is valid on the interval $(-l, l)$ (notice this is not the same interval as the sine and cosine series),

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right), \tag{42.5}$$

the coefficients are given by

$$A_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)\,dx \qquad n = 0, 1, 2, \ldots \tag{42.6a}$$

$$B_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)\,dx \qquad n = 1, 2, \ldots) \tag{42.6b}$$

Let's do another example of computing Fourier series before we begin discussing Fourier series convergence.

Example 42.1. *Compute the Fourier series for* $f(x) = \begin{cases} 2 & -2 \le x < -1 \\ 1 - x & -1 \le x < 2 \end{cases}$ *on the interval* $(-2, 2)$.

We start by using the Euler-Fourier formulas (42.6). For the cosine terms, (42.6a) gives

$$\begin{aligned} A_0 &= \frac{1}{2}\int_{-2}^{2} f(x)\,dx \\ &= \frac{1}{2}\left(\int_{-2}^{-1} 2\,dx + \int_{-1}^{2} 1 - x\,dx\right) \\ &= \frac{1}{2}\left(2 + \frac{3}{2}\right) = \frac{7}{4} \end{aligned}$$

and

$$\begin{aligned}
A_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)\,dx \\
&= \frac{1}{2}\left(\int_{-2}^{-1} 2\cos\left(\frac{n\pi x}{2}\right)\,dx + \int_{-1}^{2}(1-x)\cos\left(\frac{n\pi x}{2}\right)\,dx\right) \\
&= \frac{1}{2}\left(\frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{-1} + \frac{2(1-x)}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_{-1}^{2} - \frac{4}{n^2\pi^2}\left(\cos\left(\frac{n\pi x}{2}\right)\Big|_{-1}^{2}\right)\right) \\
&= \frac{1}{2}\left(-\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right)\right) \\
&= \begin{cases} \frac{2}{n^2\pi^2} & n \text{ odd} \\ 0 & n = 4m \\ -\frac{4}{n^2\pi^2} & n = 4m+2 \end{cases}.
\end{aligned}$$

Also, (42.6b) gives

$$\begin{aligned}
B_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)\,dx \\
&= \frac{1}{2}\left(\int_{-2}^{-1} 2\sin\left(\frac{n\pi x}{2}\right)\,dx + \int_{-1}^{2}(1-x)\sin\left(\frac{n\pi x}{2}\right)\,dx\right) \\
&= \frac{1}{2}\left(\frac{-4}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{-1} - \frac{2(1-x)}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_{-1}^{2} - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\Big|_{-1}^{2}\right) \\
&= \frac{1}{2}\left(\frac{6}{n\pi}\cos(n\pi) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right) \\
&= \begin{cases} \frac{3}{n\pi} & n \text{ even} \\ -\frac{3}{n\pi} - \frac{2}{n^2\pi^2} & n = 4m+1 \\ -\frac{3}{n\pi} + \frac{2}{n^2\pi^2} & n = 4m+3 \end{cases}.
\end{aligned}$$

So we have

$$\begin{aligned}
f(x) = \frac{7}{8} + \sum_{m=1}^{\infty} & \frac{2}{(4m+1)^2\pi^2}\cos\left(\frac{(4m+1)\pi x}{2}\right) + \left(-\frac{3}{(4m+1)\pi} - \frac{2}{(4m+1)^2\pi^2}\right)\sin\left(\frac{(4m+1)\pi x}{2}\right) \\
&- \frac{4}{(4m+2)^2\pi^2}\cos\left(\frac{(4m+2)\pi x}{2}\right) + \frac{3}{(4m+2)\pi}\sin\left(\frac{(4m+2)\pi x}{2}\right) \\
&+ \frac{2}{(4m+3)^2\pi^2}\cos\left(\frac{(4m+3)\pi x}{2}\right) + \left(-\frac{3}{(4m+3)\pi} + \frac{2}{(4m+3)^2\pi^2}\right)\sin\left(\frac{(4m+3)\pi x}{2}\right) \\
&+ \frac{3}{4m\pi}\sin\left(\frac{4m\pi x}{2}\right).
\end{aligned}$$

□

This example served as sort of a "worst case scenario" example: there are a lot of Fourier coefficients to keep track of. Notice that for each value of $m$, though, the summand specifies four different Fourier terms, for $4m$, $4m+1$, $4m+2$, and $4m+3$. This can often happen; in some cases (depending on $l$), even more will be required.

## 1. Convergence of Fourier Series

So, we know that if a function $f(x)$ is to have a Fourier series on an appropriate interval, the coefficients have to be given by (42.2) for a sine series (42.1) on $(0, l)$, (42.4) for a cosine series (42.3) on $(0, l)$, and (42.6) for a full series (42.5) on $(-l, l)$. But what, if anything, do these series converge to? We'll think about Fourier convergence with respect to the full Fourier series first, and then see how this relates to the sine and cosine series.

First, we have to require that $f(x)$ is *piecewise smooth*. This is stronger than the notion of piecewise continuity that we saw earlier when discussing Laplace transforms; we need to be able to divide up $(-l, l)$ into a finite number of subintervals so that both $f(x)$ and its derivative $f'(x)$ are continuous on each interval. Moreover, the only discontinuities we allow for either (at the boundary points of the subintervals) are jump discontinuities.

Example 42.2. Any continuous function with continuous derivative on the desired interval is automatically piecewise smooth. □

Example 42.3. Consider the function from Example 42.1,

$$f(x) = \begin{cases} 2 & -2 \leq x < -1 \\ 1 - x & -1 \leq x \leq 2 \end{cases}.$$

$f(x)$ is continuous for all $x$ in $(-2, 2)$, but the derivative $f'(x)$ has a discontinuity at $x = -1$. however, this is a jump discontinuity, with $\lim_{x \to -1^-} f'(x) = 0$ and $\lim_{x \to -1^+} = -1$. Thus $f(x)$ is piecewise smooth. □

The next thing to notice is that, even though we only need $f(x)$ to be defined on $(-l, l)$ to compute its Fourier series, the Fourier series itself is defined for all $x$. Moreover, all of the terms in a Fourier series are $2l$-periodic; they're all either constants or of the form $\sin\left(\frac{n\pi x}{l}\right)$ or $\cos\left(\frac{n\pi x}{l}\right)$. So we can regard the Fourier series either as the expansion of a function on $(-l, l)$ or as the expansion of a $2l$-periodic function on $-\infty < x < \infty$.

What will this $2l$-periodic function be? It will have to coincide on $(-l, l)$ with $f(x)$ (as it is also the expansion of just $f(x)$ on that interval) and still be $2l$-periodic. We define the *periodic extension* of $f(x)$ to be

$$f_{\text{per}}(x) = f(x - 2lm) \qquad \text{for } -l + 2lm < x < l + 2lm \tag{42.7}$$

for all integers $m$.

Remark. The definition (42.7) does not specify what the periodic extension is at the endpoints $x = l + 2lm$. This is because the extension will, in general, have jumps at these points; this happens whenever $f(-l^+) \neq f(l^-)$.

Now we can say what the Fourier series of $f(x)$ converges to[1].

Theorem 42.1 (Fourier Series Convergence). *Suppose $f(x)$ is piecewise smooth on $(-l, l)$. Then, at $x = x_0$, the Fourier series of $f(x)$ will converge to*

- *$f_{per}(x_0)$ if $f_{per}$ is continuous at $x_0$ or*
- *the average of the one-sided limits $\frac{1}{2}\left[f_{per}\left(x_0^+\right) + f_{per}\left(x_0^-\right)\right]$ if $f_{per}$ has a jump discontinuity at $x = x_0$.*

Theorem 42.1 tells us that, in particular, on the interval $(-l, l)$ the Fourier series will *almost* converge to the original function $f(x)$, with the only problems occurring at the discontinuities.

[1] If you haven't noticed by now, I have no reluctance whatsoever about ending sentences with prepositions.

Example 42.4. *What does the Fourier series of* $f(x) = \begin{cases} 1 & -3 \le x \le 0 \\ 2x & 0 < x \le 3 \end{cases}$ *converge to at* $x = -2$, $x = 0$, $x = 3$, $x = 5$, *and* $x = 6$*?*

The first two points are inside the original interval of definition of $f(x)$, so we can just directly consider that instead of having to consider $f_{\text{per}}(x)$. The only discontinuity of $f(x)$ occurs at $x = 0$. So at $x = -2$, $f(x)$ is nice and continuous, so the Fourier series will converge to $f(-2) = 1$. On the other hand, at $x = 0$ we have a jump discontinuity, so the Fourier series will converge to the average of the one-sided limits. $f(0^+) = \lim_{x\to 0^+} = 0$ while $f(0^-) = \lim_{x\to 0^-} = 1$, so the Fourier series will converge to $\frac{1}{2}\left[f(0^+) + f(0^-)\right] = \frac{1}{2}$.

What happens at the other three points? Here we have to consider $f_{\text{per}}(x)$ and where it has jump discontinuities. These can only occur either at $x = x_0 + 2lm$ where $-l < x_0 < l$ is a jump discontinuity of $f(x)$ or at endpoints $x = \pm l + 2lm$, since the periodic extension might not "sync up" at these points, producing a jump discontinuity.

At $x = 3$, we're at one of these "boundary points," and the left-sided limit is 6 while the right-sided limit is 1. Thus the Fourier series will converge here to $\frac{7}{2}$. $x = 5$, on the other hand, is a point of continuity for $f_{\text{per}}(x)$, and so the Fourier series will converge to $f_{\text{per}}(5) = f(-1) = 1$. $x = 6$, though, is a jump discontinuity (corresponding to $x = 0$), and so the Fourier series will converge to $\frac{1}{2}$. □

Example 42.5. *Where does the Fourier series for* $f(x) = \begin{cases} 2 & -2 \le x < -1 \\ 1 - x & -1 \le x \le 2 \end{cases}$ *converge at* $x = -7$, $x = -1$, *and* $x = 6$*?*

Our first observation is that there are no points inside $(-2, 2)$ where $f(x)$ is discontinuous. Thus the only points where the periodic extension might be discontinuous are the "boundary points" $x = \pm 2 + 4k$. In fact, as $f(-2) \ne f(2)$, these will be points of discontinuity. So, $f_{\text{per}}(x)$ is continuous at $x = -7$, as it's not one of these boundary points and we have $f_{\text{per}}(-7) = f(1) = 0$, which is what the Fourier series will converge to. The same goes for $x = -1$: the Fourier series will converge to $f(-1) = 2$.

On the other hand, $x = 6$ is one of these endpoint jump discontinuities. The left-sided limit is -1, while the right-sided limit is 2, so the Fourier series will converge to their average, $\frac{1}{2}$. □

## 2. Sine and Cosine Series

Before we can apply the discussion from Section 1 to Fourier sine and cosine series, we need to review some facts about even and odd functions.

**2.1. Even and Odd Functions.** Recall that an *even* function is a function satisfying

$$g(-x) = g(x).$$

This means that the graph $y = g(x)$ is symmetric with respect to the $y$-axis. An *odd* function satisfies

$$g(-x) = -g(x),$$

meaning that its graph $y = g(x)$ is symmetric with respect to the origin.

Example 42.6. A monomial $x^n$ is even if $n$ is even and odd if $n$ is odd. $\cos(x)$ is even while $\sin(x)$ is odd. $\tan(x)$ is odd. □

There are some rules for how products and sums behave (none are especially difficult: just think about how many negatives you have popping out of the arguments and canceling).

- If $g(x)$ is odd and $h(x)$ is even, their product $g(x)h(x)$ is odd.
- If $g(x)$ and $h(x)$ are either both even or both odd, $g(x)h(x)$ is even.
- The sums of two even functions or two odd functions is again even or odd, respectively.

- The sum of an even and an odd function can be anything. In fact, any function on $(-l, l)$ can be written as a sum of an even function, called the *even part*, and an odd function, called the *odd part*.
- Differentiation and integration change the parity of a function. That is, if $f(x)$ is even, $\frac{df}{dx}$ and $\int_0^x f(s)\,ds$ are both odd, and vice versa.

The graph of an odd function $g(x)$ must pass through the origin by its definition. This also tells us that if $g(x)$ is even, so long as $g'(0)$ exists, $g'(0) = 0$.

Definite integrals on symmetric intervals of odd and even functions have very useful properties.

$$\int_{-l}^{l} (odd)\,dx = 0 \qquad \text{and} \qquad \int_{-l}^{l} (even)\,dx = 2\int_{0}^{l} (even)\,dx \tag{42.8}$$

Given a function $f(x)$ defined on $(0, l)$, there is only one way to extend it to $(-l, l)$ to an even or odd function. The *even extension* of $f(x)$ is

$$f_{\text{even}}(x) = \begin{cases} f(x) & \text{for } 0 < x < l \\ f(-x) & \text{for } -l < x < 0. \end{cases} \tag{42.9}$$

This is just its reflection across the $y$-axis. Notice that the even extension is not necessarily defined at the origin.

The *odd extension* of $f(x)$ is

$$f_{\text{odd}}(x) = \begin{cases} f(x) & \text{for } 0 < x < l \\ -f(-x) \text{for } -l < x < 0 & \\ 0 & \text{for } x = 0. \end{cases} \tag{42.10}$$

This is just its reflection through the origin.

**2.2. Fourier Sine Series.** Each of the terms in the Fourier sine series for $f(x)$ (42.1), $\sin\left(\frac{n\pi x}{l}\right)$, is odd. As with the full Fourier series, each of these terms also has period $2l$. So we can think of the Fourier sine series as the expansion of an odd function with period $2l$ defined on the entire line which coincides with $f(x)$ on $(0, l)$.

In fact, (**??**) can be used to show that the full Fourier series of $f_{\text{odd}}(x)$ is precisely the same as the Fourier sine series of $f(x)$. Let

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right)$$

be the Fourier series for $f_{\text{odd}}(x)$, with coefficients given by (42.6).

$$A_n = \frac{1}{l}\int_{-l}^{l} f_{\text{odd}}(x)\cos\left(\frac{n\pi x}{l}\right)\,dx$$

But $f_{\text{odd}}$ is odd and cos is even, so their product is again odd.

$$= 0$$

$$B_n = \frac{1}{l}\int_{-l}^{l} f_{\text{odd}}(x)\sin\left(\frac{n\pi x}{l}\right)\,dx$$

But both $f_{\text{odd}}$ and sin are odd, so their product is even.

$$= \frac{2}{l}\int_0^l f_{\text{odd}}(x)\sin\left(\frac{n\pi x}{l}\right)\,dx$$
$$= \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)\,dx,$$

which are just the Fourier sine coefficients of $f(x)$. Thus, as the Fourier sine series of $f(x)$ is the full Fourier series of $f_{\text{odd}}(x)$, the $2l$-periodic odd function that the Fourier sine series expands is just the periodic extension of $f_{\text{odd}}$ as given in (42.7).