LECTURE 41

Fourier Series

Last lecture, we saw that solutions to the heat equation

$$u_t = ku_{xx},$$

$0 < x < l$, $t > 0$, with homogeneous Dirichlet conditions $u(0,t) = u(l,t) = 0$, had the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right), \tag{41.1}$$

while the heat equation with homogeneous Neumann conditions $u_x(0,t) = u_x(l,t) = 0$ had solutions of the form

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \cos\left(\frac{n\pi x}{l}\right). \tag{41.2}$$

For this to make sense given some initial condition $u(x,0) = f(x)$, (41.1) says that we need to be able to write

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \tag{41.3}$$

for some coefficients A_n, while (41.2) requires us to have

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) \tag{41.4}$$

for appropriate coefficients. Expressions of the form (41.3) are called the *Fourier sine series* of $f(x)$ and expressions of the form (41.4) are called the *Fourier cosine series* of $f(x)$.

There are two important issues here which we will need to discuss.

- Is it possible to find appropriate coefficients for series of the form (41.3) and (41.4) for a given $f(x)$?
- For which $f(x)$ will the Fourier series converge, if any? And what exactly do they converge to?

We'll leave the issue of convergence for a later discussion. First, let's try to find the coefficients.

1. The Euler-Fourier Formula

Fortunately, there is a very beautiful and conceptual formula for the Fourier coefficients, called the *Euler-Fourier formula*.

1.1. Fourier sine series.

We start off with the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right).$$

How can we find the coefficients A_n? The key observation is that sine functions have the property that

$$\int_0^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right)\, dx = 0 \tag{41.5}$$

if $m \neq n$ are both positive integers. This can be seen by direct integration. We start by recalling the trig identity

$$\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b).$$

Then the integral in (41.5) equals

$$\frac{l}{2(m-n)\pi}\sin\left(\frac{(m-n)\pi x}{l}\right)\Bigg|_0^l - \frac{l}{2(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{l}\right)\Bigg|_0^l$$

so long as $m \neq n$. But these terms are just linear combinations of $\sin((m \pm n)\pi)$ and $\sin(0)$, and so everything vanishes.

Now, let's fix m and multiply (41.3) by $\sin\left(\frac{m\pi x}{l}\right)$. Integrating term by term[1], we get

$$\begin{aligned}\int_0^l f(x)\sin\left(\frac{m\pi x}{l}\right)\,dx &= \int_0^l \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\,dx \\ &= \sum_{n=1}^{\infty} A_n \int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\,dx.\end{aligned}$$

Due to (41.5), all but one of these terms vanishes, namely the one with $n = m$. So all we have left is

$$\int_0^l f(x)\sin\left(\frac{m\pi x}{l}\right)\,dx = A_m \int_0^l \sin^2\left(\frac{m\pi x}{l}\right) = \frac{1}{2} l A_m$$

and so

$$A_m = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{m\pi x}{l}\right)\,dx. \tag{41.6}$$

To summarize what we're computed: if $f(x)$ has a Fourier sine expansion (41.3), the coefficients must be given by (41.6). These are the only possible coefficients for such a series; however, we haven't yet shown that (41.3) is a valid expression for $f(x)$.

EXAMPLE 41.1. *Compute a Fourier sine series for $f(x) = 1$ on $0 \leq x \leq l$.*

By (41.6), the coefficients must be given by

$$\begin{aligned}A_m &= \frac{2}{l}\int_0^l \sin\left(\frac{m\pi x}{l}\right)\,dx \\ &= -\frac{2}{m\pi}\cos\left(\frac{m\pi x}{l}\right)\Bigg|_0^l \\ &= \frac{2}{m\pi}(1-\cos(m\pi)) = \frac{2}{m\pi}\left(1-(-1)^m\right).\end{aligned}$$

So we have $A_m = \frac{4}{m\pi}$ if m is odd and $A_m = 0$ if m is even. Thus on $(0, l)$,

$$\begin{aligned}1 &= \frac{4}{\pi}\left(\sin\left(\frac{\pi x}{l}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{l}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{l}\right) + \dots\right) \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi x}{l}\right).\end{aligned}$$

□

[1] I know, I know...we can't always integrate infinite series term by term. We can in this case, though. Trust me.

Example 41.2. *Compute a Fourier sine series for $f(x) = x$ on $0 \le x \le l$.*
In this case, (41.6) yields a formula for the coefficients of

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l x \sin\left(\frac{m\pi x}{l}\right)\,dx \\ &= -\frac{2x}{m\pi}\cos\left(\frac{m\pi x}{l}\right) + \frac{2l}{m^2\pi^2}\sin\left(\frac{m\pi x}{l}\right)\Big|_0^l \\ &= -\frac{2l}{m\pi}\cos(m\pi) + \frac{2l}{m^2\pi^2}\sin(m\pi) \\ &= (-1)^{m+1}\frac{2l}{m\pi}. \end{aligned}$$

So on $(0,l)$, we have

$$\begin{aligned} x &= \frac{2l}{\pi}\left(\sin\left(\frac{\pi x}{l}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{l}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{l}\right) - \dots\right) \\ &= \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi x}{l}\right) - \frac{1}{2n}\sin\left(\frac{2n\pi x}{l}\right). \end{aligned}$$

□

1.2. Fourier cosine series. Now, let's consider the Fourier cosine series

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{l}\right).$$

Here we have the fact (computed just as for (41.5)), that

$$\int_0^l \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)\,dx = 0. \tag{41.7}$$

Exercise. Verify (41.7).

By the exact same computation as for sines, but with sines replaced by cosines, if $m \neq 0$ we get

$$\int_0^l f(x)\cos\left(\frac{m\pi x}{l}\right)\,dx = A_m\int_0^l \cos^2\left(\frac{m\pi x}{l}\right)\,dx = \frac{1}{2}lA_m.$$

If $m = 0$, we have

$$\int_0^l f(x)\cdot 1\,dx = \frac{1}{2}A_0\int_0^l 1^2\,dx = \frac{1}{2}lA_0.$$

Thus, for all $m > 0$, we have

$$A_m = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{m\pi x}{l}\right)\,dx. \tag{41.8}$$

This is why we put the $\frac{1}{2}$ explicitly out front of the constant A_0 in (41.4).

Example 41.3. *Compute a Fourier cosine series for $f(x) = 1$ on $0 \le x \le l$.*
By (41.8), the coefficients are given by

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l \cos\left(\frac{m\pi x}{l}\right)\,dx \\ &= \frac{2}{m\pi}\sin\left(\frac{m\pi x}{l}\right)\Big|_0^l \\ &= \frac{2}{m\pi}\sin(m\pi) = 0. \end{aligned}$$

So the only nonzero coefficient we have occurs for the constant term A_0, and this Fourier cosine series is then trivial,

$$1 = 1 + 0\cos\left(\frac{\pi x}{l}\right) + 0\cos\left(\frac{2\pi x}{l}\right) + \dots$$

This should make sense: the Fourier cosine expansion is unique, and the above sum is obvious. □

EXAMPLE 41.4. *Compute a Fourier cosine series for $f(x) = x$.*

(41.8) becomes, for $m \neq 0$,

$$\begin{aligned}
A_m &= \frac{2}{l}\int_0^l x\cos\left(\frac{m\pi x}{l}\right)\,dx \\
&= \frac{2x}{m\pi}\sin\left(\frac{m\pi x}{l}\right) + \frac{2l}{m^2\pi^2}\cos\left(\frac{m\pi x}{l}\right)\Big|_0^l \\
&= \frac{2l}{m\pi}\sin(m\pi) + \frac{2l}{m^2\pi^2}(\cos(m\pi) - 1) \\
&= \frac{2l}{m^2\pi^2}\left((-1)^m - 1\right) \\
&= \begin{cases} -\frac{4l}{m^2\pi^2} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}.
\end{aligned}$$

If $m = 0$, we have

$$A_0 = \frac{2}{l}\int_0^l x\,dx = l.$$

So on $(0, l)$, we have the Fourier cosine series

$$\begin{aligned}
x &= \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\left(\frac{\pi x}{l}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{l}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{l}\right) + \dots\right) \\
&= \frac{l}{2} + \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{l}\right).
\end{aligned}$$

□

1.3. Full Fourier series. The full Fourier series, or just the *Fourier series*, of $f(x)$ on the interval $-l < x < l$, is defined as

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{l}\right) + B_n\sin\left(\frac{n\pi x}{l}\right). \tag{41.9}$$

REMARK. Be careful: now the interval we're working on is twice as long.

The computation of coefficients for the formulas is analogous to that for the Fourier sine series (41.3) and the Fourier cosine series (41.4). We need the following set of identities:

$$\begin{aligned}
&\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\,dx = 0 && \text{for all } n, m \\
&\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)\,dx = 0 && \text{for } n \neq m \\
&\int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\,dx = 0 && \text{for } n \neq m
\end{aligned}$$

$$\int_{-l}^{l} 1\cdot\cos\left(\frac{n\pi x}{l}\right)\,dx = 0 = \int_{-l}^{l} 1\cdot\sin\left(\frac{n\pi x}{l}\right)\,dx.$$

Thus, using the same procedure as for the sine and cosine series, we can get the coefficients; we'll fix m and multiply by $\cos\left(\frac{m\pi x}{l}\right)$, and do the same for $\sin\left(\frac{m\pi x}{l}\right)$. So we need to calculate the integrals of the squares

$$\int_{-l}^{l} \cos^2\left(\frac{n\pi x}{l}\right)\,dx = 1 = \int_{-l}^{l} \sin^2\left(\frac{n\pi x}{l}\right)\,dx \qquad \text{and} \qquad \int_{-l}^{l} 1^2\,dx = 2l.$$

Exercise. Verify these integrals.

So we get the formulas

$$A_m = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{m\pi x}{l}\right)\,dx \qquad (n = 0, 1, 2, \ldots) \tag{41.10a}$$

$$B_m = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)\,dx \qquad (n = 1, 2, \ldots) \tag{41.10b}$$

for the coefficients of the full Fourier series. Notice that (41.10a) is exactly the same as (41.8) and (41.10b) is the same as (41.6).

Example 41.5. *Compute the Fourier series of $f(x) = 1 + x$.* Using (41.10), we have

$$\begin{aligned}
A_0 &= \frac{1}{l}\int_{-l}^{l}(1+x)\,dx = 2\\
A_m &= \frac{1}{l}\int_{-l}^{l}(1+x)\cos\left(\frac{m\pi x}{l}\right)\,dx\\
&= \frac{1+x}{m\pi}\sin\left(\frac{m\pi x}{l}\right) + \frac{l}{m^2\pi^2}\cos\left(\frac{m\pi x}{l}\right)\Bigg|_{-l}^{l}\\
&= \frac{l}{m^2\pi^2}(\cos(m\pi) - \cos(-m\pi)) = 0 \qquad m \neq 0\\
B_m &= \frac{1}{l}\int_{-l}^{l}(1+x)\sin\left(\frac{m\pi x}{l}\right)\,dx\\
&= -\frac{1+x}{m\pi}\cos\left(\frac{m\pi x}{l}\right) + \frac{l}{m^2\pi^2}\sin\left(\frac{m\pi x}{l}\right)\Bigg|_{-l}^{l}\\
&= -\frac{2l}{m\pi}\cos(m\pi) = (-1)^{m+1}\frac{2l}{m\pi}.
\end{aligned}$$

So the full Fourier series of $f(x)$ is

$$\begin{aligned}
1 + x &= 1 + \frac{2l}{\pi}\left(\sin\left(\frac{\pi x}{l}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{l}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{l}\right) - \ldots\right)\\
&= 1 + \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi x}{l}\right) - \frac{1}{2n}\sin\left(\frac{2n\pi x}{l}\right).
\end{aligned}$$

□