LECTURE 8

# Exact Equations

The final category of first order differential equations we will consider are the so-called *exact equations*. These nonlinear equations have the form

$$\boxed{M(x,y) + N(x,y)\frac{dy}{dx} = 0} \tag{8.1}$$

where $y = y(x)$ is a function of $x$ and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

where these two derivatives are *partial derivatives*, discussed more below.

We'll begin this lecture by briefly discussing how to compute the partial derivatives of a multivariable function. Then we'll see a motivating example of how to solve an exact equation, which will in turn lead to a discussion of why what we did in the example worked. Finally, we'll see some examples done with the solution method in mind.

## 1. Multivariable Differentiation

Suppose we have a function $f(x, y)$ depending on two variables. What does it mean to differentiate this function?

Recall that if we have a function of a single variable $g(x)$, we can interpret the derivative as the rate of change of the output of $g(x)$ as $x$ increases. We'd like to have a similar notion of differentiation for our multivariable function $f(x, y)$.

The situation is, of course, complicated. We no longer just have to worry about what happens as $x$ increases; we have to worry about how to measure the rate of change of $f(x, y)$ as $y$ increases or even as $x$ and $y$ vary in different ways. Geometrically, we can think of the graph of $f(x, y)$ as specifying some surface in three-dimensional space, so talking about "the slope" at a point is vague in this way.

So what do we do? It turns out (and this is a topic that is explored more in a multivariable calculus class) that if we can account for how $f(x, y)$ varies as only $x$ or only $y$ vary, we don't have to worry about mixtures of the two. Let's consider how we want to express the derivative of $f(x, y)$ at a point $(x_0, y_0)$. Without loss of any generality, we can consider what happens only as $x$ varies (since the $y$ situation will be analogous, and we've already punted on considering mixing the two to a class that is more suited to discussing that). Fixing $y = y_0$ reduces our function of two variables $f(x, y)$ to a function of a single variable $g(x) = f(x, y_0)$. So we define the *partial derivative of $f$ with respect to $x$ at* $(x_0, y_0)$, denoted by

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0),$$

to be $g'(x_0)$ with $g(x) = f(x, y_0)$. Geometrically, this is the slope of the curve on the graph of $f$ corresponding to the line $y = y_0$ at $x = x_0$.

What does this mean, computationally? Our function $g$ is defined by treating $y$ as the constant $y_0$. So its derivative will be equivalent to the derivative of $f$ taken with respect to $x$ while treating $y$ as a constant. The following examples should clarify the above discussion.

Example 8.1. Let $f(x,y) = x^2y + y^2$. Then

$$\frac{\partial f}{\partial x} = 2xy$$
$$\frac{\partial f}{\partial y} = x^2 + 2y.$$

□

Example 8.2. Let $f(x,y) = y\sin(x)$.

$$\frac{\partial f}{\partial x} = y\cos(x)$$
$$\frac{\partial f}{\partial y} = \sin(x).$$

□

We will also need to be able to recognize the multivariable chain rule. The relevant version of this says that if we have a function $\Phi(x, y(x))$ depending on some variable $x$ and a function $y$ depending on $x$, then

(8.2) $$\frac{d\Phi}{dx} = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{dy}{dx} = \Phi_x + \Phi_y y'.$$

## 2. Exact Equations

Before we talk in more detail about how to solve an exact equation, let's work one example to get a feel for what an exact equation is and why the solution method works. We'll rework this later after a more general discussion to reveal all the details.

Example 8.3. Consider

$$2xy - 9x^2 + \left(2y + x^2 + 1\right)\frac{dy}{dx} = 0.$$

The first step in solving an exact equation is to find a certain function $\Phi(x,y)$. We'll see how to find this function later (and in fact this computation is where most of the work lies), but for this example it will turn out that the desired function is

$$\Phi(x,y) = y^2 + \left(x^2 + 1\right)y - 3x^3.$$

Notice that if we compute the partial derivatives of $\Phi$, we obtain

$$\Phi_x(x,y) = 2xy - 9x^2$$
$$\Phi_y(x,y) = 2y + x^2 + 1.$$

Looking back at the differential equation, we can see that it can be rewritten as

$$\Phi_x + \Phi_y\frac{dy}{dx} = 0.$$

Then, thinking back to the chain rule as expressed in Equation 8.2, we see that the differential equation is just

$$\frac{d\Phi}{dx} = 0.$$

This tells us that our function $\Phi$ must be equal to some constant, since its ordinary derivative is zero, and thus the general solution is

$$y^2 + \left(x^2 + 1\right)y - 3x^3 = c$$

for some constant $c$, which is just the constant of integration. If we had an initial condition, at this point we could solve for $c$ and get the particular solution to the initial value problem. □

Let's think about what we saw in the previous example. An exact equation has the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

with $M_y(x,y) = N_x(x,y)$. The key to finding the solution to an exact equation is to construct a function $\Phi(x,y)$ such that the differential equation turns into

$$\frac{d\Phi}{dx} = 0$$

by using the multivariable chain rule (8.2), as we did above. Thus we require that $\Phi$ satisfy

$$\Phi_x(x,y) = M(x,y)$$
$$\Phi_y(x,y) = N(x,y).$$

Remark. One of the standard facts of multivariable calculus is that mixed partial derivatives commute. This is why we want $M_y = N_x$: $M_y = \Phi_{xy}$ and $N_x = \Phi_{yx}$, and these should be identical if such a function $\Phi$ can exist. It's imperative to check that a function is actually exact (by computing $M_x$ and $N_y$ and seeing that they coincide) before proceeding with the solution process, or there's no way it can work.

Once we have this function $\Phi$, then we know $\frac{d\Phi}{dx} = 0$, and hence

$$\Phi(x,y) = c,$$

yielding an implicit general solution to the differential equation.

So, we can see that the real work involves computing $\Phi(x,y)$. How can we do it? Let's revisit Example 8.3, this time filling in all the details. We'll also add an initial condition into the mix.

Example 8.4. *Solve the initial value problem*

$$2xy - 9x^2 + \left(2y + x^2 + 1\right)\frac{dy}{dx} = 0 \qquad y(0) = 2$$

.

Let's begin, as we always should, be checking that this differential equation is actually exact. Comparing the equation to Equation 8.1, we have

$$M(x,y) = 2xy - 9x^2$$

and

$$N(x,y) = 2y + x^2 + 1.$$

Then $M_y = 2x$ and $N_x = 2x$, hence the equation is exact.

Now, how do we find $\Phi(x,y)$? We have that $\Phi_x = M$ and $\Phi_Y = N$. Thus we could compute $\Phi$ in one of two ways:

$$\Phi(x,y) = \int M\,dx \qquad \text{or} \qquad \Phi(x,y) = \int N\,dy.$$

We'll need to be slightly careful here, as we'll see, but first let's just note that it doesn't usually matter which of $M$ or $N$ you choose to integrate to get $\Phi$. There are some examples in which one is clearly easier, but it's a judgement call: integrate whichever seems easier to you. In this case, they're equally easy, so let's use the first one.

$$\Phi(x,y) = \int 2xy - 9x^2\,dx = x^2y - 3x^3 + h(y).$$

Notice the presence of the function $h(y)$ in the integral. This is the equivalent of the constant of integration that we obtain when we integrate a single variable function, since if we differentiate $\Phi$ with respect to $x$, any function that just depends on $y$ will vanish. So instead of tacking on a constant to the end, we need to have a function of $y$.

If we had integrated $N$ with respect to $y$ to get $\Phi$, we would have needed an equivalent function of $x$, something like $h(x)$, to play the role of the "constant of integration." It's very important that this not be forgotten.

Now all we need to do is to find $h(y)$ and we'll have our $\Phi$. How do we do this? We know that if we differentiate $\Phi$ with respect to $x$, $h$ will vanish, which is utterly unhelpful. However, if we differentiate $\Phi$ with respect to $y$, we're in good shape, since $h'$ will hang around and we know that $\Phi_y = N$. So we'll compute $\Phi_y$, and any terms in $N$ that aren't in $\Phi_y$ must be $h'(y)$. Analogously, if we had started by integrating $N$ with respect to $y$ to find $\Phi$, we would want to differentiate $\Phi$ with respect to $x$ and compare with $M$.

Ok, so we can see that $\Phi_y = x^2 + h'(y)$ and $N = x^2 + 2y + 1$. So since these are equal, we must have $h'(y) = 2y + 1$, and so

$$h(y) = \int h'(y)\,dy = y^2 + y$$

.

Remark. As we've tended to do, we're going to be a little careless with constants. In general, we'll drop the constant of integration from our computation of $h$, since it will end up combining with the constant $c$ that we get as part of the solution process.

Thus, we have

$$\Phi(x, y) = x^2y - 3x^3 + y^2 + y = y^2 + (x^2 + 1)y - 3x^3,$$

which is precisely the $\Phi$ that we used in Example 8.3. From here on, it's exactly the same as it was there: we observe that the differential equation is just

$$\frac{d\Phi}{dx} = 0$$

and thus $\Phi(x, y) = y^2 + (x^2 + 1)y - 3x^3 = c$ for some constant $c$. To compute $c$, we'll use our initial condition $y(0) = 2$.

$$2^2 + 2 = c \Rightarrow c = 6$$

and so we have a particular solution of

$$y^2 + (x^2 + 1)y - 3x^3 = 6.$$

This is a quadratic equation in $y$, so as we've seen we can complete the square or use the quadratic formula to get an explicit solution, which we want to do when we can.

$$\begin{aligned}
y^2 + (x^2 + 1)y - 3x^3 &= 6 \\
y^2 + (x^2 + 1)y + \frac{(x^2 + 1)^2}{4} &= 6 + 3x^3 + \frac{(x^2 + 1)^2}{4} \\
\left(y + \frac{x^2 + 1}{2}\right)^2 &= \frac{x^4 + 12x^3 + 2x^2 + 25}{4} \\
y(x) &= \frac{-(x^2 + 1) \pm \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}
\end{aligned}$$

Now we use the initial condition to figure which whether we want the + or the − in that ±. Since $y(0) = 2$, we have

$$2 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = 2, -3.$$

Thus we see that we'll want the + solution, and our particular solution is

$$y(x) = \frac{-(x^2 + 1) + \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}$$

□

The following examples should be less long, since we won't need to repeat the rationales behind each of the steps.

Example 8.5. *Solve the initial value problem*

$$2xy^2 + 2 = 2(3 - x^2 y)y' \qquad y(-1) = 1.$$

First, we need to put the equation into the form of Equation 8.1:

$$2xy^2 + 2 - 2(3 - x^2 y)y' = 0.$$

Now, we have $M(x, y) = 2xy^2 + 2$ and $N(x, y) = -2(3 - x^2 y)$ (incorporating the minus into $N$ is very important: otherwise, the derivatives won't work out). Thus $M_y = 4xy = N_x$ and the equation is exact.

The next step is to compute $\Phi(x, y)$. In this example, it's equally easy to integrate either $M$ or $N$, so let's use $N$:

$$\Phi(x, y) = \int N\, dy = \int 2x^2 y - 6\, dy = x^2 y^2 - 6y + h(x).$$

To find $h(x)$, we compute $\Phi_x = 2xy^2 + h'(x)$ and notice that for this to be equal to $M$, $h'(x) = 2$. Hence $h(x) = 2x$ and we have an implicit solution of

$$x^2 y^2 - 6y + 2x = c.$$

Now, we use the initial condition $y(-1) = 1$:

$$1 - 6 - 2 = c \Rightarrow c = -7.$$

So our implicit solution is

$$x^2 y^2 - 6y + 2x + 7 = 0.$$

Again, we can solve for the explicit solution by completing the square or using the quadratic formula:

$$\begin{aligned} y(x) &= \frac{6 \pm \sqrt{36 - 4x^2(2x + 7)}}{2x^2} \\ &= \frac{3 \pm \sqrt{9 - 2x^3 - 7x^2}}{x^2} \end{aligned}$$

and using the initial condition, we see we want the "-" solution, so the explicit particular solution is

$$y(x) = \frac{3 - \sqrt{9 - 2x^3 - 7x^2}}{x^2}.$$

□

Example 8.6. *Solve the IVP*

$$\frac{2ty}{t^2 + 1} - 2t - \left(4 - \ln\left(t^2 + 1\right)\right) y' = 0 \qquad y(2) = 0$$

*and find the solution's interval of validity.*

This is already in the form of Equation 8.1, so let's start by checking if it's exact. $M(t, y) = \frac{2ty}{t^2+1} - 2t$ and $N(t, y) = \ln(t^2 + 1) - 4$, so $M_y = \frac{2t}{t^2+1} = N_t$. Thus the equation is exact. Now, let's compute $\Phi$. In this case, it will be easiest to integrate $M$:

$$\Phi = \int M\, dt = \int \frac{2ty}{t^2 + 1} - 2t\, dt = y \ln(t^2 + 1) - t^2 + h(y).$$

$$\Phi_y = \ln(t^2 + 1) + h'(y) = \ln(t^2 + 1) - 4 = N$$

so we conclude $h'(y) = -y$ and thus $h(y) = -4y$. So our implicit solution is then

$$y\ln(t^2+1) - t^2 - 4y = c$$

and we use our initial condition to compute $c = -4$. Thus the particular solution is

$$y\ln(t^2+1) - t^2 - 4y = -4,$$

and this is very easy to solve explicitly. Doing so, we obtain

$$y(x) = \frac{t^2-4}{\ln(t^2+1)-4}.$$

Now, let's find the interval of validity. We don't have to worry about the logarithm since $t^2+1>0$ for all $t$. Thus we only need to avoid division by zero, so we need to avoid the following points.

$$\begin{aligned} \ln(t^2+1) - 4 &= 0 \\ \ln(t^2+1) &= 4 \\ t^2 &= e^4 - 1 \\ t &= \pm\sqrt{e^4-1} \end{aligned}$$

So there are three possible intervals of validity:

$$(-\infty, -\sqrt{e^4-1}, (-\sqrt{e^4-1}, \sqrt{e^4-1}), and(\sqrt{e^4-1}, \infty).$$

The middle one contains $t = 2$, so our interval of validity is $(-\sqrt{e^4-1}, \sqrt{e^4-1})$. □

Example 8.7. *Solve*

$$3y^3e^{3xy} - 1 + \left(2ye^{3xy} + 3xy^2e^{3xy}\right)y' = 0 \qquad y(1) = 2.$$

We have $M(x,y) = 3y^3e^{3xy} - 1$ and $N(x,y) = 2ye^{3xy} + 3xy^2e^{3xy}$, so

$$M_y = 9y^2e^{3xy} + 9xy^3e^{3xy} = N_x.$$

Thus the equation is exact. We'll integrate $M$, since it's a bit easier.

$$\Phi = \int M\,dx = \int 3y^3e^{3xy} - 1 = y^2e^{3xy} - x + h(y)$$

and

$$\Phi_y = 2ye^{3xy} + 3xy^2e^{3xy} + h'(y).$$

Comparing $\Phi_y$ to $N$, we see that they are already identical, so we must have $h'(y) = 0$ and hence $h(y) = 0$ (since we're ignoring constants in $h$). So we have

$$y^2e^{3xy} - x = c,$$

and using the initial condition gives $c = 4e^6 - 1$. Thus our implicit particular solution is

$$y^2e^{3xy} - x = 4e^6 - 1,$$

and we're done because we won't be able to solve this explicitly. □