

LECTURE 32

More Linear Algebra

1. Back to Systems

We return our attention now to the system of equations

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\&\vdots \\a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n\end{aligned}\tag{32.1}$$

To express this system of equations in matrix form, we start by writing both sides as vectors.

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Notice that the left hand side of the equation can be rewritten as a matrix-vector product.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

We can simplify this notation by writing

$$\mathbf{Ax} = \mathbf{b}\tag{32.2}$$

where \mathbf{x} is the vector whose entries are the variables in the system, A is the matrix of coefficients of the system (called the *coefficient matrix*), and \mathbf{b} is the vector whose entries are the right-hand sides of the equations. We call (32.2) the *matrix form* of the system of equations (32.1).

We know that the system of equations (32.1), and hence (32.2), have zero, one, or infinitely many solutions. Suppose $\det(A) \neq 0$, *i.e.*, A is nonsingular. Then (32.2) has only one solution, namely,

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

So we can rewrite our earlier Theorem 31.1 in the following way.

Theorem 32.1. *Given the system of equations (32.2),*

- (1) *if $\det(A) \neq 0$, there is exactly one solution;*
- (2) *if $\det(A) = 0$, there are either zero or infinitely many solutions.*

Recall that if our system (32.1) were homogeneous, *i.e.*, if each $b_i = 0$, we always have the trivial solution $x_i = 0$. Denoting the vector with entries all 0 by $\mathbf{0}$, the matrix form of a homogeneous system is

$$A\mathbf{x} = \mathbf{0}. \quad (32.3)$$

Thus we can express the earlier Theorem 31.2 as follows.

Theorem 32.2. *Given the homogeneous system of equations (32.3),*

- (1) *if $\det(A) \neq 0$, there is exactly one solution, $\mathbf{x} = \mathbf{0}$;*
- (2) *if $\det(A) = 0$, there will be infinitely many nonzero solutions.*

2. Eigenvalues and Eigenvectors

We'll now need a slight digression to one of the most important aspects of linear algebra. We've already observed that if we multiply a vector by a matrix, we get another vector, *i.e.*,

$$A\boldsymbol{\eta} = \mathbf{y}.$$

A natural question to ask is when \mathbf{y} is just a scalar multiple of $\boldsymbol{\eta}$; in other words, for what vectors $\boldsymbol{\eta}$ is multiplication by A equivalent to a "stretching" of $\boldsymbol{\eta}$, or, more formally, when do we have

$$A\boldsymbol{\eta} = \lambda\boldsymbol{\eta}? \quad (32.4)$$

If (32.4) is satisfied for some constant λ and some vector $\boldsymbol{\eta}$, we call $\boldsymbol{\eta}$ an *eigenvector* of A with *eigenvalue* λ . We can first notice that if $\boldsymbol{\eta} = \mathbf{0}$, (32.4) will be satisfied for any λ . We're not generally interested in a trivial solution like this, though, so we will always assume $\boldsymbol{\eta} \neq \mathbf{0}$.

So how can we find solutions to (32.4)? Let's start by rewriting it, recalling that I is the 2×2 identity matrix.

$$\begin{aligned} A\boldsymbol{\eta} &= \lambda\boldsymbol{\eta} \\ A\boldsymbol{\eta} - \lambda I\boldsymbol{\eta} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{\eta} &= \mathbf{0} \end{aligned}$$

We had to multiply λ by the identity I before we could factor it out. This is because we can't subtract a constant from a matrix, so we had to "convert," in a way, the constant λ into a matrix by multiplying it by I .

So, we've turned (32.4) into the equation

$$(A - \lambda I)\boldsymbol{\eta} = \mathbf{0}, \quad (32.5)$$

which is the matrix form for a homogeneous system of equations. By Theorem 32.3 from last class, if $A - \lambda I$ is nonsingular, the only solution is the trivial solution $\boldsymbol{\eta} = \mathbf{0}$, which we've already said we're not interested in. On the other hand, if $A - \lambda I$ is singular, we'll have infinitely many nonzero solutions. Thus the condition that we'll need to find any eigenvalues and eigenvectors that may exist for A is for

$$\det(A - \lambda I) = 0.$$

It's a basic fact that this equation, $\det(A - \lambda I) = 0$, is an n^{th} degree polynomial if A is an $n \times n$ matrix. It's called the *characteristic equation* of the matrix A .

As a result, the Fundamental Theorem of Algebra tells us that an $n \times n$ matrix A has n eigenvalues, counting multiplicities. To find them, all we have to do is to find the roots of an n^{th} degree polynomial, which is no problem for small n . Suppose we've found these eigenvalues. What can we conclude about their associated eigenvectors?

We call k vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ linearly independent if the only constants c_1, c_2, \dots, c_k satisfying

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_k\mathbf{x}_k = \mathbf{0}$$

are $c_1 = c_2 = \dots = c_k = 0$. This definition should look similar; it's entirely analogous to our earlier definition of linear independence for functions. Now, we have the following fact.

Theorem 32.3. *If $\lambda_1, \lambda_2, \dots, \lambda_n$ is the complete list of eigenvalues of A , including multiplicities, then*

- (1) *if λ occurs only once in the list, it is called simple;*
- (2) *if λ occurs $k > 1$ times, it has multiplicity k ;*
- (3) *if $\lambda_1, \lambda_2, \dots, \lambda_k$ ($k \leq n$) are the simple eigenvalues of A with corresponding eigenvectors $\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \dots, \boldsymbol{\eta}^{(k)}$, then these eigenvectors $\boldsymbol{\eta}^{(i)}$ are linearly independent;*
- (4) *if λ is an eigenvalue with multiplicity k , then λ will have anywhere from 1 to k linearly independent eigenvectors.*

This fact should look familiar from our discussion of second and higher order equations; we had a similar result about roots of the characteristic equation in that case. This result tells us when we have linearly independent eigenvectors, which is handy when trying to solve systems of differential equations.

Now, once we have the eigenvalues, how do we find their associated eigenvectors? Let's do a couple of examples to see.

Example 32.1. *Find the eigenvalues and eigenvectors of*

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}.$$

The first thing we need to do is to find the roots of the characteristic equation of the matrix

$$A - \lambda I = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 - \lambda & 4 \\ 2 & 1 - \lambda \end{pmatrix}.$$

This is

$$0 = \det(A - \lambda I) = (3 - \lambda)(1 - \lambda) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1).$$

Thus the two eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -1$.

Now, to find the eigenvectors we need to plug each eigenvalue into (32.5) and solve for $\boldsymbol{\eta}$.

(1) $\lambda_1 = 5$:

In this case, (32.5) becomes the following system.

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \boldsymbol{\eta} = \mathbf{0}$$

Next, we'll write out the components of the two vectors and multiply through.

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2\eta_1 + 4\eta_2 \\ 2\eta_1 - 4\eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For this vector equation to hold, the components must match up. So we've got to find a solution to the system

$$\begin{aligned} -2\eta_1 + 4\eta_2 &= 0 \\ 2\eta_1 - 4\eta_2 &= 0. \end{aligned}$$

Notice that these are the same equation: the second line only differs from the first by multiplication by a constant, in this case -1 . This will always be the case if we've found our eigenvalues correctly, since we know that $A - \lambda I$ is singular and so our system (32.5) should have infinitely many solutions. In fact, going back to our matrix $A - \lambda I$, we could notice there that the rows only differed by a constant factor. This is a good place to check that our earlier algebra was correct: if the rows differ by more than just a constant factor, something's gone wrong.

Since these equations are identical, we can just choose one (whichever is convenient works fine) and obtain a relation between the eigenvector components η_1 and η_2 . Let's choose the first. This gives

$$2\eta_1 = 4\eta_2,$$

and so we have $\eta_1 = 2\eta_2$. As a result, any eigenvector corresponding to $\lambda_1 = 5$ has the form

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2\eta_2 \\ \eta_2 \end{pmatrix}.$$

There are infinitely many vectors of this form, of course; we only need one. We can select one by choosing a value for η_2 . The only restriction is that we shouldn't take $\eta_2 = 0$, since then $\boldsymbol{\eta} = \mathbf{0}$, which we don't want. We may choose, for example, $\eta_2 = 1$, and then we have

$$\boldsymbol{\eta}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(2) $\lambda_2 = -1$:

In the previous case, we went into more detail than we will in future examples. The process is exactly the same, however.

Plugging in λ_2 into (32.5) gives the system

$$\begin{aligned} \begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 4\eta_1 + 4\eta_2 \\ 2\eta_1 + 2\eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

The two equations corresponding to this vector equation are

$$\begin{aligned} 4\eta_1 + 4\eta_2 &= 0 \\ 2\eta_1 + 2\eta_2 &= 0. \end{aligned}$$

Once again, these only differ by a constant factor: the first equation is twice the second. Let's choose the second equation to work with, though in this case it doesn't matter at all; both are equally easy. We have

$$\eta_1 = -\eta_2,$$

and so any eigenvector has the form

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix}.$$

We can once again choose $\eta_2 = 1$, giving us a second eigenvector of

$$\boldsymbol{\eta}^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Summarizing, the eigenvalue/eigenvector pairs of A are

$$\begin{aligned} \lambda_1 &= 5 & \boldsymbol{\eta}^{(1)} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \lambda_2 &= -1 & \boldsymbol{\eta}^{(2)} &= \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \end{aligned}$$

□

Remark. We could have ended up with any number of different vectors for our eigenvectors $\boldsymbol{\eta}^{(1)}$ and $\boldsymbol{\eta}^{(2)}$, depending on the choices we made at the end. However, they would have only differed by a multiplicative constant, and this is ok.

Example 32.2. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix}.$$

The characteristic equation for this matrix is

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 2 - \lambda & -1 \\ 5 & 4 - \lambda \end{vmatrix} \\ &= (2 - \lambda)(4 - \lambda) + 5 \\ &= \lambda^2 - 6\lambda + 13. \end{aligned}$$

Completing the square (or using the quadratic formula), we see that the roots are $r_{1,2} = 3 \pm 2i$. If we get complex eigenvalues, to find the eigenvectors we proceed just as we did in the previous example.

$$(1) \lambda_1 = 3 + 2i$$

Here the matrix equation

$$(A - \lambda I)\boldsymbol{\eta} = \mathbf{0}$$

becomes

$$\begin{aligned} \begin{pmatrix} -1 - 2i & -1 \\ 5 & 1 - 2i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} (-1 - 2i)\eta_1 - \eta_2 \\ 5\eta_1 + (1 - 2i)\eta_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

So the pair of equations we get are

$$\begin{aligned} (-1 - 2i)\eta_1 - \eta_2 &= 0 \\ 5\eta_1 + (1 - 2i)\eta_2 &= 0. \end{aligned}$$

It's not as obvious as in the last example, but these two equations are scalar multiples: if we multiply the first equation by $-1 + 2i$, we recover the second. Now, we choose one of these equations to work with. Let's use the first. This gives us that $\eta_2 = (-1 - 2i)\eta_1$, so any eigenvector has the form

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ (-1 - 2i)\eta_1 \end{pmatrix}.$$

Choosing $\eta_1 = 1$ gives a first eigenvector of

$$\boldsymbol{\eta}^{(1)} = \begin{pmatrix} 1 \\ -1 - 2i \end{pmatrix}.$$

(2) $\lambda_1 = 3 - 2i$

Here the matrix equation

$$(A - \lambda I)\boldsymbol{\eta} = \mathbf{0}$$

becomes

$$\begin{pmatrix} -1 + 2i & -1 \\ 5 & 1 + 2i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} (-1 + 2i)\eta_1 - \eta_2 \\ 5\eta_1 + (1 + 2i)\eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So the pair of equations we get are

$$\begin{aligned} (-1 + 2i)\eta_1 - \eta_2 &= 0 \\ 5\eta_1 + (1 + 2i)\eta_2 &= 0. \end{aligned}$$

Let's use the first equation again. This gives us that $\eta_2 = (-1 + 2i)\eta_1$, so any eigenvector has the form

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ (-1 + 2i)\eta_1 \end{pmatrix}.$$

Choosing $\eta_1 = 1$ gives a second eigenvector of

$$\boldsymbol{\eta}^{(2)} = \begin{pmatrix} 1 \\ -1 + 2i \end{pmatrix}.$$

To sum, A has the following eigenvalue/eigenvector pairs.

$$\begin{aligned} \lambda_1 = 3 - 2i & \quad \boldsymbol{\eta}^{(1)} = \begin{pmatrix} 1 \\ -1 - 2i \end{pmatrix} \\ \lambda_2 = 3 + 2i & \quad \boldsymbol{\eta}^{(2)} = \begin{pmatrix} 1 \\ -1 + 2i \end{pmatrix} \end{aligned}$$

□

Remark. Notice that the eigenvalues came in *complex conjugate* pairs; that is, they were of the form $a \pm bi$. This is always the case for complex roots, as we can easily see from the quadratic formula (we also saw this in the sections on second and higher order equations). Moreover, the complex entries in the eigenvectors were also complex conjugate, and the real entries were the same (all up to a multiplicative constant, of course). This is always the case so long as A doesn't have any complex entries.