LECTURE 29

The Dirac Delta And Laplace Transforms

A few lectures ago, we discussed step (or Heaviside) functions, which we could think of as a "switch," changing the nonhomogeneous term at specified times. From an applications viewpoint, we could interpret the presence of a step function as a new external force which would be applied to the system starting at a certain point.

But what if, instead of having a new force which would persist for the rest of the system's use, all we wanted to do was apply a "large" force over a very "small" time frame? Instead of turning on a new voltage in an electrical circuit, we might want to model a short. A mechanical interpretation might be the use of a hammer to strike an object or the striking of a baseball with a bat. It would be nice if we had a mathematical way of representing these types of forces.

To this end, we will introduce a new "function", the Dirac[1] delta "function."

1. The Dirac Delta

There are several ways to define the Dirac delta, but we will do so by requiring it to satisfy several properties.

Definition 29.1. The Dirac delta at $t = c$, denoted $\delta(t-c)$, satisfies the following properties:

(1) $\delta(t-c) = 0 \qquad t \neq c;$

(2) $\displaystyle\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,dt = 1 \qquad \text{for any } \epsilon > 0;$

(3) $\displaystyle\int_{c-\epsilon}^{c+\epsilon} f(t)\delta(t-c)\,dt = f(c) \qquad \text{for any } \epsilon > 0.$

Heuristically, we can think of $\delta(t-c)$ as having an "infinite" value at $t = c$, so that its total energy is 1, all concentrated at that point. So the Dirac delta can be thought of as an instantaneous impulse at time $t = c$. Notice that the second and third properties work when the limits are the endpoints of any interval including $t = c$.

This should sound like a very odd function to you. It's zero everywhere but at one point, and yet the integral of it is 1. This is why I put "function" in quotation marks earlier: the Dirac delta is not really a function, but is instead an example of something called a *generalized function* or a *distribution*. However, it's perfect for our purposes: it does a fantastic job of modeling a sudden shock to a system.

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[1] Paul Dirac (1902-1984) was a British physicist who helped found quantum mechanics. Dirac was well regarded for his mathematical proficiency, which led him to make connections between certain physical principles and mathematical formalisms. Dirac proposed the "Dirac equation" as a relativistic equation of motion for an electron's wavefunction. This led to him predicting the existence of the positron, the antiparticle of the electron. Dirac's work also led to the way for Feynman's introduction of the path integral formulation of quantum mechanics. Dirac's books on general relativity and quantum mechanics are excellent introductory textbooks to those subjects.

On a lighter note, when Dirac was asked about what he thought of poetry, he answered "In science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before. But in poetry, it's the exact opposite."

2. Laplace Transform of the Dirac Delta

Before we try solving initial value problems involving the Dirac delta, we will need to know its Laplace transform. By definition,

$$\mathcal{L}\left\{\delta(t-c)\right\} = \int_0^\infty e^{-st}\delta(t-c)\,dt = e^{-cs}$$

by the third property of the Dirac delta. Notice that this requires $c > 0$, since otherwise the integral in question will just vanish.

Now, let's try solving some initial value problems involving the Dirac delta.

Example 29.2. *Solve the following initial value problem.*

$$y'' + 3y' - 10y = 4\delta(t-2) \qquad y(0) = 2 \quad y'(0) = -3$$

We begin, as usual, by taking the Laplace transform of the entire equation.

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) - 10Y(s) &= 4e^{-2s} \\ (s^2 + 3s - 10)Y(s) - 2s - 3 &= 4e^{-2s} \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{4e^{-2t}}{(s+5)(s-2)} + \frac{2s+3}{(s+5)(s-2)} \\ &= Y_1(s)e^{-2t} + Y_2(s) \end{aligned}$$

We'll leave it to you to verify that the partial fractions of each of the pieces are

$$\begin{aligned} Y_1(s) &= \frac{4}{(s+5)(s-2)} = \frac{4}{7}\frac{1}{s-2} - \frac{4}{7}\frac{1}{s+5} \\ Y_2(s) &= \frac{2s+3}{(s+5)(s-2)} = \frac{1}{s-2} + \frac{1}{s+5}. \end{aligned}$$

Thus they have the inverse transforms

$$\begin{aligned} y_1(t) &= \frac{4}{7}e^{2t} - \frac{4}{7}e^{-5t} \\ y_2(t) &= e^{2t} + e^{-5t} \end{aligned}$$

and the solution is then

$$\begin{aligned} y(t) &= y_1(t-2)u_2(t) + y_2(t) \\ &= u_2(t)\left(\frac{4}{7}e^{2(t-2)} - \frac{4}{7}e^{-5(t-2)}\right) + e^{2t} + e^{-5t} \\ &= u_2(t)\left(\frac{4}{7}e^{2t-4} - \frac{4}{7}e^{-5t+10}\right) + e^{2t} + e^{-5t}. \end{aligned}$$

□

Notice that, even though the exponential in the transform $Y(s)$ came originally from the delta function, once we inverse transformed the corresponding term it became a step function. This will be the case in general; we shouldn't be surprised, because it turns out the there's a relationship between the step function $u_c(t)$ and the delta $\delta(t-c)$.

We begin with the integral

$$\begin{aligned} \int_{-\infty}^{t} \delta(u-c)\,du &= \begin{cases} 0 & t < c \\ 1 & t > c \end{cases} \\ &= u_c(t) \end{aligned}$$

Then the Fundamental Theorem of Calculus says

$$u_c'(t) = \frac{d}{dt}\left(\int_{-\infty}^{t} \delta(u-c)\,du\right) = \delta(t-c).$$

Thus the Dirac delta at $t = c$ is actually the derivative of the step function at $t = c$, which we can think of geometrically by remembering that the graph of $u_c(t)$ is horizontal at every $t \neq c$, hence at those points $t = 0$, and it has a jump of one at $t = c$.

Let's do another couple of examples of initial value problems involving the Dirac delta.

EXAMPLE 29.3. *Solve the following initial value problem.*

$$y'' + 4y' + 9y = 2\delta(t-1) + e^t \qquad y(0) = 0 \quad y'(0) = -1$$

First, we Laplace transform both sides and solve for $Y(s)$.

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 9Y(s) &= 2e^{-s} + \frac{1}{s-1} \\ (s^2+4s+9)Y(s) + 1 &= 2e^{-s} + \frac{1}{s-1} \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{2e^{-s}}{s^2+4s+9} + \frac{1}{(s-1)(s^2+4s+9)} - \frac{1}{s^2+4s+9} \\ &= Y_1(s)e^{-s} + Y_2(s) - Y_3(s) \end{aligned}$$

Next, we have to prepare $Y(s)$ for the inverse transform. This will require completing the square for $Y_1(s)$ and $Y_3(s)$, while we'll need to first partial fraction $Y_2(s)$. We'll leave the details to you to verify, but, to get everything in the correct form for inverse transforming, we obtain

$$\begin{aligned} Y_1(s) &= \frac{2}{s^2+4s+9} = \frac{2}{(s+2)^2+5} \\ &= \frac{2}{\sqrt{5}}\frac{\sqrt{5}}{(s+2)^2+5} \\ Y_2(s) &= \frac{1}{(s-1)(s^2+4s+9)} = \frac{1}{14}\left(\frac{1}{s-1} - \frac{s+5}{(s+2)^2+5}\right) \\ &= \frac{1}{14}\left(\frac{1}{s-1} - \frac{(s+2-2)+5}{(s+2)^2+5}\right) \\ &= \frac{1}{14}\left(\frac{1}{s-1} - \frac{s+2}{(s+2)^2+5} - \frac{3}{(s+2)^2+5}\right) \\ &= \frac{1}{14}\left(\frac{1}{s-1} - \frac{s+2}{(s+2)^2+5} - \frac{3}{\sqrt{5}}\frac{\sqrt{5}}{(s+2)^2+5}\right) \\ Y_3(s) &= \frac{1}{(s^2+4s+9)} = \frac{1}{(s+2)^2+5} \\ &= \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{(s+2)^2+5}. \end{aligned}$$

So their inverse transforms are

$$\begin{aligned}
y_1(t) &= \frac{2}{\sqrt{5}}e^{-2t}\sin\left(\sqrt{5}t\right) \\
y_2(t) &= \frac{1}{14}\left(e^t - e^{-2t}\cos\left(\sqrt{5}t\right) - \frac{3}{\sqrt{5}}e^{-2t}\sin\left(\sqrt{5}t\right)\right) \\
y_3(t) &= \frac{1}{\sqrt{5}}e^{-2t}\sin\left(\sqrt{5}t\right).
\end{aligned}$$

Thus, since our original transformed function was

$$Y(s) = Y_1(s)e^{-s} + Y_2(s) - Y_3(s),$$

we obtain

$$\begin{aligned}
y(t) &= u_1(t)y_1(t-1) + y_2(t) - y_3(t) \\
&= u_1(t)\left(\frac{2}{\sqrt{5}}e^{-2t+2}\sin\left(\sqrt{5}t - \sqrt{5})\right)\right) \\
&\quad + \frac{1}{14}\left(e^t - e^{-2t}\cos\left(\sqrt{5}t\right) - \frac{3}{\sqrt{5}}e^{-2t}\sin\left(\sqrt{5}t\right)\right) \\
&\quad - \frac{1}{\sqrt{5}}e^{-2t}\sin\left(\sqrt{5}t\right).
\end{aligned}$$

□

EXAMPLE 29.4. *Solve the following initial value problem.*

$$y'' + 16y = 2u_3(t) + 5\delta(t-1) \qquad y(0) = 1 \quad y'(0) = 2$$

Again, we begin by taking the Laplace transform of the entire equation and applying our initial conditions, then solving for $Y(s)$.

$$\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + 16Y(s) &= \frac{2e^{-3s}}{s} + 5e^{-s} \\
(s^2+16)Y(s) - s - 2 &= \frac{2e^{-3s}}{s} + 5e^{-s}
\end{aligned}$$

$$\begin{aligned}
Y(s) &= \frac{2e^{-3s}}{s(s^2+16)} + \frac{5e^{-s}}{s^2+16} + \frac{s+2}{s^2+16} \\
&= Y_1(s)e^{-3s} + Y_2(s)e^{-s} + Y_3(s)
\end{aligned}$$

The only one of these three functions that needs partial fractioning[2] is the first one. The rest can be dealt with directly; all they need is a little modification. We end up with

$$\begin{aligned}
Y_1(s) &= \frac{2}{s(s^2+16)} = \frac{1}{8}\frac{1}{s} - \frac{1}{8}\frac{s}{s^2+16} \\
Y_2(s) &= \frac{5}{s^2+16} = \frac{5}{4}\frac{4}{s^2+16} \\
Y_3(s) &= \frac{s+2}{s^2+16} = \frac{s}{s^2+16} + \frac{1}{2}\frac{4}{s^2+16}
\end{aligned}$$

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[2] Verbification of nouns is fun, easy, and useful!

and so the associated inverse transforms are

$$\begin{aligned}
y_1(t) &= \frac{1}{8} - \frac{1}{8}\cos(4t) \\
y_2(t) &= \frac{5}{4}\sin(4t) \\
y_3(t) &= \cos(4t) + \frac{1}{2}\sin(4t).
\end{aligned}$$

Our solution is the inverse transform of

$$Y(s) = Y_1(s)e^{-3s} + Y_2(s)e^{-s} + Y_3(s),$$

and this will be

$$\begin{aligned}
y(t) &= u_3(t)y_1(t-3) + u_1(t)y_2(t-1) + y_3(t) \\
&= u_3(t)\left(\frac{1}{8} - \frac{1}{8}\cos(4(t-3))\right) + \frac{5}{4}u_1(t)\sin(4(t-1)) + \cos(4t) + \frac{1}{2}\sin(4t) \\
&= u_3(t)\left(\frac{1}{8} - \frac{1}{8}\cos(4t-12)\right) + \frac{5}{4}u_1(t)\sin(4t-4) + \cos(4t) + \frac{1}{2}\sin(4t).
\end{aligned}$$

□