LECTURE 11

Complex and Repeated Roots

1. Complex Roots

Last time, we saw that when the characteristic equation has complex roots $r_{1.2} = \alpha \pm i\beta$, the general solution of

$$ay'' + by' + cy = 0$$

is

$$y(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t).$$

Let's do some examples.

EXAMPLE 11.1. *Solve the IVP*

$$y'' - 4y' + 9y = 0 \qquad y(0) = 0, y'(0) = -2.$$

The characteristic equation is

$$r^2 - 4r + 9 = 0$$

which has roots $r_{1,2} = 2 \pm i\sqrt{5}$. Thus the general solution and its derivative are

$$\begin{aligned} y(t) &= c_1 e^{2t}\cos(\sqrt{5}t) + c_2 e^{2t}\sin(\sqrt{5}t) \\ y'(t) &= 2c_1 e^{2t}\cos(\sqrt{5}t) - \sqrt{5}c_1 e^{2t}\sin(\sqrt{5}t) + 2c_2 e^{2t}\sin(\sqrt{5}t) + \sqrt{5}c_2 e^{2t}\cos(\sqrt{5}t). \end{aligned}$$

If we apply the initial conditions, we get

$$\begin{aligned} 0 &= c_1 \\ -2 &= 2c_1 + \sqrt{5}c_2 \end{aligned}$$

which is solved by $c_1 = 0$ and $c_2 = -\frac{2}{\sqrt{5}}$. So the particular solution is

$$y(t) = -\frac{2}{\sqrt{5}} e^{2t}\sin(\sqrt{5}t).$$

□

EXAMPLE 11.2. *Solve the IVP*

$$y'' - 8y' + 17y = 0 \qquad y(0) = 2, y'(0) = 5.$$

The characteristic equation is

$$r^2 - 8r + 17 = 0,$$

which has roots $r_{1,2} = 4 \pm i$. Hence the general solution and its derivative are

$$\begin{aligned} y(t) &= c_1 e^{4t}\cos(t) + c_2 e^{4t}\sin(t) \\ y'(t) &= 4c_1 e^{4t}\cos(t) - c_1 e^{4t}\sin(t) + 4c_2 e^{4t}\sin(t) + c_2 e^{4t}\cos(t) \end{aligned}$$

and plugging in initial conditions yields the system

$$\begin{aligned} 2 &= c_1 \\ 5 &= 4c_1 + c_2, \end{aligned}$$

so we conclude $c_1 = 2$ and $c_2 = -3$ and the particular solution is

$$y(t) = 2e^{4t}\cos(t) - 3e^{4t}\sin(t).$$

$\square$

Example 11.3. *Solve the IVP*

$$4y'' + 12y' + 10y = 0 \qquad y(0) = -1, y'(0) = 3.$$

The characteristic equation is

$$4r^2 + 12r + 10 = 0,$$

which has roots $r_{1,2} = -\frac{3}{2} \pm \frac{1}{2}i$. So the general solution and its derivative are

$$\begin{aligned} y(t) &= c_1 e^{\frac{3}{2}t}\cos\left(\frac{t}{2}\right) + c_2 e^{\frac{3}{2}t}\sin\left(\frac{t}{2}\right) \\ y'(t) &= \frac{3}{2}c_1 e^{\frac{3}{2}t}\cos\left(\frac{t}{2}\right) - \frac{1}{2}c_1 e^{\frac{3}{2}t}\sin\left(\frac{t}{2}\right) + \frac{3}{2}c_2 e^{\frac{3}{2}t}\sin\left(\frac{3}{2}\right) + \frac{1}{2}c_2 e^{\frac{3}{2}t}\cos\left(\frac{t}{2}\right). \end{aligned}$$

Plugging in the initial conditions yields

$$\begin{aligned} -1 &= c_1 \\ 3 &= \frac{3}{2}c_1 + \frac{1}{2}c_2 \end{aligned}$$

which has solution $c_1 = -1$ and $c_2 = 9$. The particular solution is

$$y(t) = -e^{\frac{3}{2}t}\cos\left(\frac{t}{2}\right) + 9e^{\frac{3}{2}t}\sin\left(\frac{t}{2}\right).$$

$\square$

Example 11.4. *Solve the IVP*

$$y'' + 4y = 0 \qquad y\left(\frac{\pi}{4}\right) = -10, y'\left(\frac{\pi}{4}\right) = 4.$$

The characteristic equation is

$$r^2 + 4 = 0,$$

which has roots $r_{1,2} = \pm 2i$. The general solution and its derivative are

$$\begin{aligned} y(t) &= c_1\cos(2t) + c_2\sin(2t) \\ y'(t) &= -2c_1\sin(2t) + 2c_2\cos(2t). \end{aligned}$$

The initial conditions gives the system

$$\begin{aligned} -10 &= c_2 \\ 4 &= -2c_1 \end{aligned}$$

so we conclude $c_1 = -2$ and $c_2 = -10$ and the particular solution is

$$y(t) = -2\cos(2t) - 10\sin(2t).$$

$\square$

2. Repeated Roots

The last case we have to consider is when the characteristic equation has a repeated root $r_1 = r_2 = r$. This is problematic, though: our usual method of finding solutions to constant coefficient equations would lead us to form the two solutions

$$y_1(t) = e^{r_1 t} = e^{rt} \qquad y_2(t) = e^{r_2 t} = e^{rt}.$$

But these are the same, and definitely not different in any sense, let alone different enough to form a general solution. So we're left needing a second solution which is "different" (in a way I promise I will make precise later) from $y_1(t) = e^{rt}$. What do we do?

Let's start by recalling that if the quadratic equation $ar^2 + br + c = 0$ has a repeated root r, it must be $r = -\frac{b}{2a}$. Thus our solution is, more precisely, $y_1(t) = e^{-\frac{b}{2a}t}$. We know that any constant multiple of y_1 is a solution. These, though, won't help us find a different enough second solution. The question that might come to mind is if it's possible that we can find a solution of the form

$$y_2(t) = v(t)y_1(t) = v(t)e^{-\frac{b}{2a}t},$$

i.e., y_2 is the product of some other function of t and y_1.

In the past, when we've guessed at the form of a solution, we've checked our guess and derived specifics by plugging the guess into the differential equation. This is no different. So we'll need to differentiate $y_2(t)$:

$$\begin{aligned} y_2'(t) &= v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v(t)e^{-\frac{b}{2a}t} \\ y_2''(t) &= v''(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t} \\ &= v''(t)e^{-\frac{b}{2a}t} - \frac{b}{a}v'(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t}. \end{aligned}$$

For the rest of this calculation, I'm going to stop explicitly denoting the parameter of v, but don't forget that v is a function, not a constant. Now, plugging in:

$$\begin{aligned} a\left(v''e^{-\frac{b}{2a}t} - \frac{b}{a}v'e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}ve^{-\frac{b}{2a}t}\right) + b\left(v'e^{-\frac{b}{2a}t} - \frac{b}{2a}ve^{-\frac{b}{2a}t}\right) + c\left(ve^{-\frac{b}{2a}t}\right) &= 0 \\ e^{-\frac{b}{2a}t}\left(av'' + (-b+b)v' + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v\right) &= 0 \\ e^{-\frac{b}{2a}t}\left(av'' - \frac{1}{4a}\left(b^2 - 4ac\right)v\right) &= 0. \end{aligned}$$

Since we're in the repeated root case, we know that the discriminant $b^2 - 4ac = 0$. As exponentials are never zero, we're left with the condition

$$av'' = 0 \Rightarrow v'' = 0.$$

We can drop the a because a can't be zero: if a were zero, our equation wouldn't be second order! So, what form can v take? It must be that v is linear, *i.e.*

$$v(t) = ct + k$$

for some constants c and k. Thus, for any such $v(t)$, $y_2(t) = v(t)e^{-\frac{b}{2a}t}$ will be a solution. The most general possible $v(t)$ that will work for us is $ct + k$. We can take $c = 1$ and $k = 0$ to get a specific v which is nice and simple, and then our second solution is

$$y_2(t) = te^{-\frac{b}{2a}t}$$

and the general solution is

$$y(t) = c_1e^{-\frac{b}{2a}t} + tc_2e^{-\frac{b}{2a}t}.$$

Remark. Here's another way of looking at this choice of constants. Suppose we don't make it. Then we have, for our general solution,

$$\begin{aligned} y(t) &= c_1 e^{-\frac{b}{2a}t} + c_2(ct+k)e^{-\frac{b}{2a}t} \\ &= c_1 e^{-\frac{b}{2a}t} + c_2 cte^{-\frac{b}{2a}t} + c_2 ke^{-\frac{b}{2a}t} \\ &= (c_1 + c_2 k)e^{-\frac{b}{2a}t} + c_2 cte^{-\frac{b}{2a}t}. \end{aligned}$$

Since c_1, c_2, c, and k are just constants, we'll just roll them together and write

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t}.$$

To summarize the previous discussion: if the characteristic equation has repeated roots $r_1 = r_2 = r$, the general solution is

$$y(t) = c_1 e^{rt} + c_2 te^{rt}.$$

Let's work some examples.