

LECTURE 33

Complex Eigenvalues

Last lecture, we looked at solutions to the equation

$$\mathbf{x}' = A\mathbf{x},$$

where the eigenvalues of the matrix A were real and distinct. What happens if they are complex?

This isn't dissimilar to the complex roots case when we were considering second order equations (for reasons we've discussed already). We still have solutions of the form

$$\mathbf{x} = \boldsymbol{\eta}e^{\lambda t}$$

, where $\boldsymbol{\eta}$ is an eigenvector of A with eigenvalue λ . However, we want real-valued solutions, which we won't have if we leave them in that form.

Our strategy will be similar in this case: we'll use Euler's formula to rewrite

$$e^{(a+ib)t} = e^{at} \cos(bt) + ie^{at} \sin(bt),$$

then we'll write out one of our solutions fully into real and imaginary parts. It will turn out that each of these parts gives us a solution, and in fact they'll also form a fundamental set of solutions. Let's illustrate this by example.

EXAMPLE 33.1. *Solve the following initial value problem.*

$$\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -2 & -3 \end{pmatrix} \mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

We begin by finding the eigenvalues of A .

$$0 = \det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 6 \\ -2 & -3 - \lambda \end{vmatrix} \\ = \lambda^2 + 3$$

Thus the two eigenvalues are $\lambda_{1,2} = \pm\sqrt{3}i$.

Next, we need to find an eigenvector. It turns out we'll only need one.

We'll use $\lambda_1 = \sqrt{3}i$.

$$(A - \sqrt{3}iI)\boldsymbol{\eta}^* = \mathbf{0} \\ \begin{pmatrix} 3 - \sqrt{3}i & 6 \\ -2 & -3 - \sqrt{3}i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The system of equations to solve is

$$(3 - \sqrt{3}i)\eta_1 + 6\eta_2 = 0 \\ -2\eta_1 + (-3 - \sqrt{3}i)\eta_2 = 0.$$

We can use either equation to find solutions, but let's use the second one. This gives $\eta_1 = \frac{1}{2}(-3 - \sqrt{3}i)\eta_2$. Thus any eigenvector has the form

$$\boldsymbol{\eta} = \begin{pmatrix} \frac{1}{2}(-3 - \sqrt{3}i)\eta_2 \\ \eta_2 \end{pmatrix},$$

and choosing $\eta_2 = 2$ yields a first eigenvector

$$\boldsymbol{\eta}^{(1)} = \begin{pmatrix} -3 - \sqrt{3}i \\ 2 \end{pmatrix}.$$

Thus we have a solution

$$\mathbf{x}_1(t) = e^{\sqrt{3}it} \begin{pmatrix} -3 - \sqrt{3}i \\ 2 \end{pmatrix}.$$

Unfortunately, this is complex-valued, and we'd like a real-valued solution. We had a similar problem back when we talked about second order linear equations. What did we do then? We used Euler's formula to expand this imaginary exponential into cosine and sine terms, then split the solution into real and imaginary parts. This then gave us the two fundamental solutions we needed.

We'll do the same thing here (in reality, doing it here is why we do it there, as we've discussed). We'll use Euler's formula to expand

$$e^{\sqrt{3}it} = \cos(\sqrt{3}t) + i \sin(\sqrt{3}t),$$

then multiply it through the eigenvector. After separating into real and complex parts using the basic matrix arithmetic operations, it'll turn out that each of these parts is a solution. More to the point, they're linearly independent and give us a fundamental set of solutions.

$$\begin{aligned} \mathbf{x}_1(t) &= \left(\cos(\sqrt{3}t) + i \sin(\sqrt{3}t) \right) \begin{pmatrix} -3 - \sqrt{3}i \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -3 \cos(\sqrt{3}t) - 3i \sin(\sqrt{3}t) - \sqrt{3}i \cos(\sqrt{3}t) + \sqrt{3} \sin(\sqrt{3}t) \\ 2 \cos(\sqrt{3}t) + 2i \sin(\sqrt{3}t) \end{pmatrix} \\ &= \left(\begin{pmatrix} -3 \cos(\sqrt{3}t) + \sqrt{3} \sin(\sqrt{3}t) \\ 2 \cos(\sqrt{3}t) \end{pmatrix} + i \begin{pmatrix} -3 \sin(\sqrt{3}t) - \sqrt{3} \cos(\sqrt{3}t) \\ 2 \sin(\sqrt{3}t) \end{pmatrix} \right) \\ &= \mathbf{u}(t) + i\mathbf{v}(t). \end{aligned}$$

As noted earlier, both $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are real-valued solutions to the differential equation (this isn't hard to check; it follows from linearity of the derivative). Moreover, they're linearly independent. Our general solution is then

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{u}(t) + c_2 \mathbf{v}(t) \\ &= c_1 \begin{pmatrix} -3 \cos(\sqrt{3}t) + \sqrt{3} \sin(\sqrt{3}t) \\ 2 \cos(\sqrt{3}t) \end{pmatrix} + c_2 \begin{pmatrix} -3 \sin(\sqrt{3}t) - \sqrt{3} \cos(\sqrt{3}t) \\ 2 \sin(\sqrt{3}t) \end{pmatrix}. \end{aligned}$$

Finally, we'll use the initial condition to get c_1 and c_2 . It says

$$\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \mathbf{x}(0) = c_1 \begin{pmatrix} -3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -\sqrt{3} \\ 0 \end{pmatrix}.$$

This translates into the system

$$\begin{aligned} -3c_1 - \sqrt{3}c_2 &= -2 \\ 2c_1 &= 4 \end{aligned} \quad \Rightarrow \quad c_1 = 2 \quad c_2 = -\frac{4}{\sqrt{3}}.$$

Hence our particular solution is

$$\mathbf{x}(t) = 2 \begin{pmatrix} -3 \cos(\sqrt{3}t) + \sqrt{3} \sin(\sqrt{3}t) \\ 2 \cos(\sqrt{3}t) \end{pmatrix} - \frac{4}{\sqrt{3}} \begin{pmatrix} -3 \sin(\sqrt{3}t) - \sqrt{3} \cos(\sqrt{3}t) \\ 2 \sin(\sqrt{3}t) \end{pmatrix}.$$

□

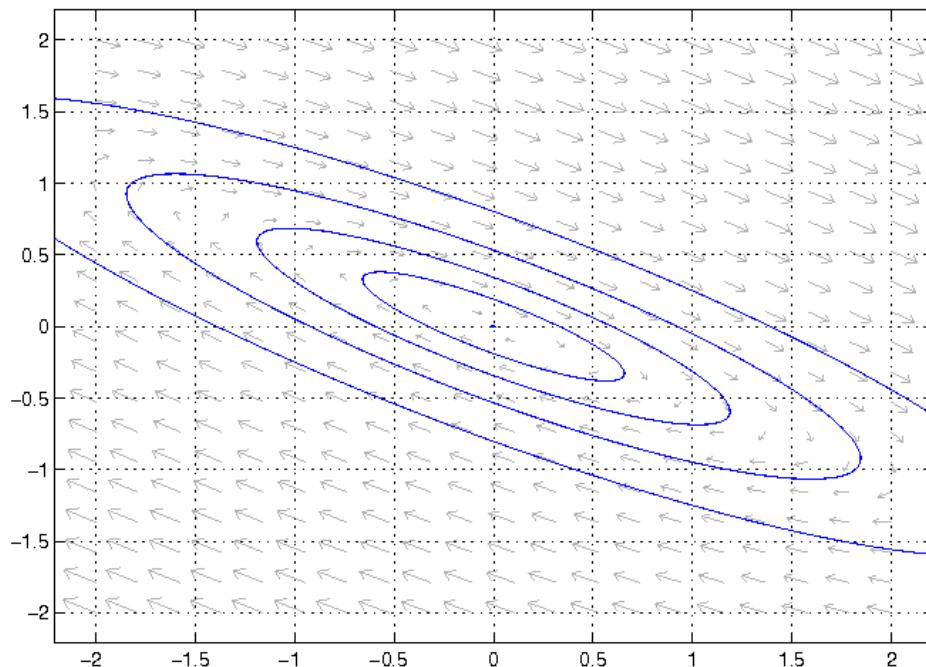


FIGURE 33.1. Phase portrait of the center point in Example 33.1.

EXAMPLE 33.2. Sketch the phase portrait of the system in Example 33.1.

The general solution to the system in Example 33.1 is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -3 \cos(\sqrt{3}t) + \sqrt{3} \sin(\sqrt{3}t) \\ 2 \cos(\sqrt{3}t) \end{pmatrix} + c_2 \begin{pmatrix} -3 \sin(\sqrt{3}t) - \sqrt{3} \cos(\sqrt{3}t) \\ 2 \sin(\sqrt{3}t) \end{pmatrix}.$$

Every term in this solution (other than the constants c_1 and c_2) is periodic; we have $\cos(\sqrt{3}t)$ and $\sin(\sqrt{3}t)$. Thus both x_1 and x_2 are periodic functions for any initial conditions. On the phase plane, this translates to trajectories which are closed; that is, they form circles or ellipses. As a result, the phase portrait looks like Figure 33.1.

This is always the case when we have purely imaginary eigenvalues, as the exponential turns into strictly a combination of sines and cosines. In this case, the equilibrium solution is called a *center* and is *neutrally stable* or just *stable* (note: not asymptotically stable).

The only work that needs to be done in these cases is to figure out the eccentricity and direction that the trajectory is traveled. The former is a bit difficult, and we usually don't care that much, but the latter is a lot easier. We can determine whether the trajectories orbit the origin in a clockwise or counterclockwise direction by calculating the tangent vector \mathbf{x}' at a single point. For example, at the point $(1, 0)$ in the previous example, we have

$$\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Thus at $(1, 0)$, the tangent vector points down and to the right. This can only happen if the trajectories circle the origin in a clockwise direction. \square

EXAMPLE 33.3. Solve the following initial value problem.

$$\mathbf{x}' = \begin{pmatrix} 6 & -4 \\ 7 & -2 \end{pmatrix} \mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

We begin by finding the eigenvalues of A .

$$0 = \det(A - \lambda I) = \begin{vmatrix} 6 - \lambda & -4 \\ 7 & -2 - \lambda \end{vmatrix} \\ = \lambda^2 - 4\lambda + 16 = (\lambda - 2)^2 + 16$$

Thus the two eigenvalues are $\lambda_{1,2} = 2 \pm 4i$.

Next, we need to find an eigenvector.

We'll use $\lambda_1 = 2 + 4i$.

$$(A - (2 + 4i)I)\boldsymbol{\eta}^* = \mathbf{0} \\ \begin{pmatrix} 4 - 4i & -4 \\ 7 & -4 - 4i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The system of equations to solve is

$$(4 - 4i)\eta_1 - 4\eta_2 = 0 \\ 7\eta_1 + (-4 - 4i)\eta_2 = 0.$$

We can use either equation to find solutions, but let's use the first one. This gives $\eta_2 = (1 - i)\eta_1$. Thus any eigenvector has the form

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ (1 - i)\eta_1 \end{pmatrix},$$

and choosing $\eta_1 = 1$ yields a first eigenvector

$$\boldsymbol{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}.$$

Thus we have a solution

$$\mathbf{x}_1(t) = e^{(2+4i)t} \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}$$

and using Euler's formula gives

$$= e^{2t} e^{4it} \begin{pmatrix} 1 \\ 1 - i \end{pmatrix} \\ = e^{2t} (\cos(4t) + i \sin(4t)) \begin{pmatrix} 1 \\ 1 - i \end{pmatrix} \\ = e^{2t} \begin{pmatrix} \cos(4t) + i \sin(4t) \\ \cos(4t) + i \sin(4t) - i \cos(4t) + \sin(4t) \end{pmatrix} \\ = e^{2t} \left(\begin{pmatrix} \cos(4t) \\ \cos(4t) + \sin(4t) \end{pmatrix} + i \begin{pmatrix} \sin(4t) \\ \sin(4t) - \cos(4t) \end{pmatrix} \right) \\ = \mathbf{u}(t) + i\mathbf{v}(t).$$

Our general solution is

$$\mathbf{x}(t) = c_1 \mathbf{u}(t) + c_2 \mathbf{v}(t) \\ = c_1 e^{2t} \begin{pmatrix} \cos(4t) \\ \cos(4t) + \sin(4t) \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \sin(4t) \\ \sin(4t) - \cos(4t) \end{pmatrix}.$$

Finally, we'll use the initial condition to get c_1 and c_2 . It says

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

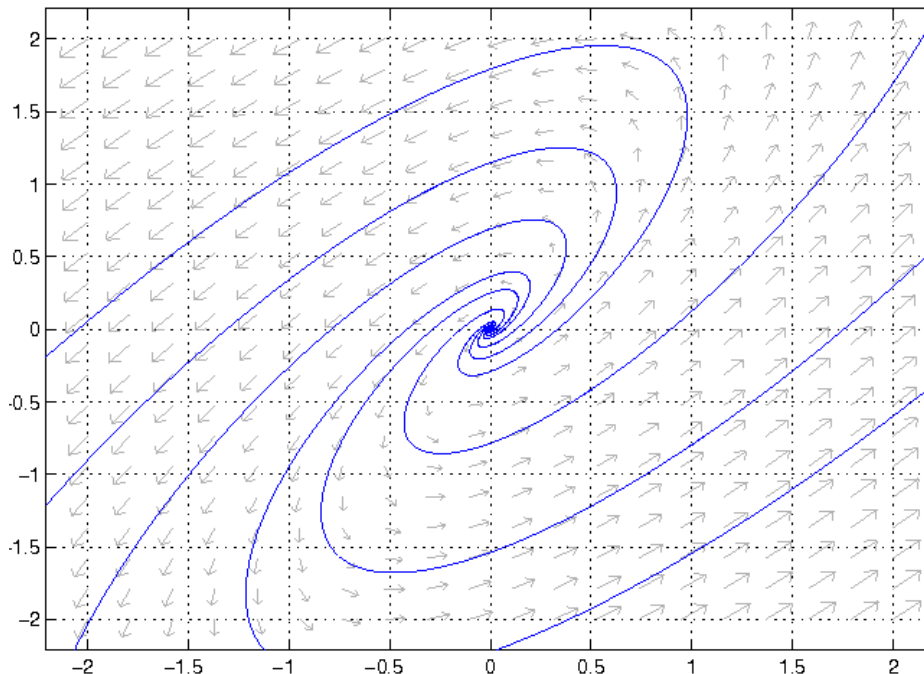


FIGURE 33.2. Phase portrait for the unstable spiral in Example 33.3.

This translates into the system

$$\begin{aligned} c_1 &= 1 \\ c_1 - c_2 &= 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ c_2 &= -2. \end{aligned}$$

Hence our particular solution is

$$\mathbf{x}(t) = e^{2t} \begin{pmatrix} \cos(4t) \\ \cos(4t) + \sin(4t) \end{pmatrix} - 2e^{2t} \begin{pmatrix} \sin(4t) \\ \sin(4t) - \cos(4t) \end{pmatrix}.$$

□

EXAMPLE 33.4. Sketch the phase portrait of the system in Example 33.3.

The only (qualitative) difference between the general solution to this example,

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} \cos(4t) \\ \cos(4t) + \sin(4t) \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \sin(4t) \\ \sin(4t) - \cos(4t) \end{pmatrix},$$

and the one in Example 33.1 is the exponential sitting out front of the periodic terms. This will make the solution quasi-periodic, rather than actually periodic. The exponential, having a positive exponent, will cause the solution to grow as $t \rightarrow \infty$ away from the origin. The solution will still rotate, however, as the trig terms will cause oscillation. Thus, rather than forming closed circles or ellipses, the trajectories will spiral out of the origin.

As a result, when we have complex (not just imaginary) eigenvalues $\lambda_{1,2} = a \pm bi$, we call the situation a *spiral*. In this case, as the real part a (which affects to the exponent) is positive, and the solution grows, the equilibrium at the center is unstable. If a had been negative, the spiral would decay into the origin, and the equilibrium would have been asymptotically stable.

So, what's there to calculate if we recognize we have a stable/unstable spiral? We still need to know the direction of rotation. This requires, as with the center, that we calculate tangent vectors

at a point or two. In this case, the tangent vector at the point $(1, 0)$ is

$$\mathbf{x}' = \begin{pmatrix} 6 & -4 \\ 7 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}.$$

Thus the tangent vector at $(1, 0)$ points up and to the right. Combined with the knowledge that the solution is leaving the origin, this can only happen if the direction of rotation of the spiral is counterclockwise. We obtain a picture as in Figure 33.2. \square