LECTURE 38

Boundary Value Problems and Eigenfunctions

1. Boundary Conditions

At this point, we're essentially finished discussing ordinary differential equations. We're going to begin discussing partial differential equations. Partial differential equations are much more complicated than ordinary differential equations: we'll need to specify the way we want solutions to behave on the boundary of the region our equation is valid on. Such data are called *boundary values* or *boundary conditions*, and a combination of a differential equation and some boundary conditions is called a *boundary value problem*.

Boundary conditions depend on the region our equation is valid on: if we have an ordinary differential equation, this will be some interval; for a partial differential equation, this might also be an interval, or it might be on a square in the plane. To try to get a better feel for how boundary values work, let's see how they affect solutions to ordinary differential equations.

Example 38.1. Let's consider the second order differential equation $y'' + y = 0$. Specifying boundary conditions for this equation involves specifying the values of the solution (or its derivatives) at two points. Let's say we fix $y(0) = 0$ and $y(2\pi) = 0$. We know that solutions to the equation have the form

$$y(x) = A\cos(x) + B\sin(x).$$

Applying the first boundary condition tell us that $0 = y(0) = A$. Applying the second condition gives $0 = y(2\pi) = B\sin(2\pi)$. Now, $\sin(2\pi) = 0$, so this condition doesn't tell us anything about B; it can be anything and this condition will still be satisfied. So the solutions to this boundary value problem are any functions of the form

$$y(x) = B\sin(x).$$

□

Example 38.2. Consider $y'' + y = 0$ with boundary values $y(0) = y(6) = 0$. This seems similar to the previous problem; the solutions have the form

$$y(x) = A\cos(x) + B\sin(x)$$

and the first condition still tells us $A = 0$. The second condition tells us that $0 = y(6) = B\sin(6)$. Now, $\sin(6) \neq 0$, so we must have $B = 0$ and the entire solution is $y(x) = 0$. □

Boundary value problems are very natural things. We can interpret Examples 38.1 and 38.2 physically. We know the equation $y'' + y = 0$ models an oscillator; something like a rock hanging from a Slinky[1] This rock oscillates with a frequency of $\frac{1}{2\pi}$. The condition $y(0) = 0$ just means that when we start observing, we want the rock to be at the equilibrium spot. If we specify $y(2\pi) = 0$, this is no problem for any solution, because it will automatically happen: the motion is 2π-periodic. On the other hand, it's impossible for the rock to return to the equilibrium point after 6 seconds. It will come back in 2π seconds, which is a bit more than 6. So the only possible way the rock can be at equilibrium after 6 seconds is if it doesn't leave, which is why the only solution is the 0 solution.

[1]It's Slinky, it's Slinky, for fun it's a wonderful toy. It's Slinky, it's Slinky, it's fun for a a girl and a boy!

Examples 38.1 and 38.2 are examples of *homogeneous* boundary value problems. We say that a boundary value problem is homogeneous if the equation is homogeneous and the two boundary conditions involve zero. That is, homogeneous boundary conditions might be of the following types:

$$\begin{aligned} y(x_1) &= 0 & y(x_2) &= 0 \\ y'(x_1) &= 0 & y(x_2) &= 0 \\ y(x_1) &= 0 & y'(x_2) &= 0 \\ y'(x_1) &= 0 & y'(x_2) &= 0. \end{aligned}$$

On the other hand, if the equation is nonhomogeneous or any of the boundary conditions don't involve zero, we say that the boundary value problem is *nonhomogeneous*. Let's look at some examples of nonhomogeneous boundary value problems.

Example 38.3. Take $y'' + 9y = 0$ with boundary conditions $y(0) = 2$ and $y\left(\frac{\pi}{6}\right) = 1$. The general solution to the differential equation is

$$y(x) = A\cos(3x) + B\sin(3x).$$

The two conditions give

$$\begin{aligned} 2 &= y(0) = A \\ 1 &= y\left(\frac{\pi}{6}\right) = B \end{aligned}$$

so that the solution is

$$y(x) = 2\cos(3x) + \sin(3x).$$

□

Example 38.4. Take $y'' + 9y = 0$ with boundary conditions $y(0) = 2$ and $y\left(2\pi\right) = 2$. The general solution to the differential equation is

$$y(x) = A\cos(3x) + B\sin(3x).$$

The two conditions give

$$\begin{aligned} 2 &= y(0) = A \\ 2 &= y\left(2\pi\right) = A. \end{aligned}$$

In other words, this time the second condition didn't give us any new information, like in Example 38.1 and B doesn't affect whether or not the solution satisfies the boundary conditions or not. We then have infinitely many solutions,

$$y(x) = 2\cos(3x) + B\sin(3x).$$

□

Example 38.5. Take $y'' + 9y = 0$ with boundary conditions $y(0) = 2$ and $y(2\pi) = 4$. The general solution to the differential equation is

$$y(x) = A\cos(3x) + B\sin(3x).$$

The two conditions give

$$\begin{aligned} 2 &= y(0) = A \\ 4 &= y\left(2\pi\right) = A. \end{aligned}$$

On the one hand, we must have $A = 2$. On the other, we have to have $A = 4$. This isn't possible, and this boundary value problem in fact has no solutions. □

These examples hopefully illustrate that a small change to the boundary conditions can dramatically change the problem, unlike with initial value problems.