LECTURE 6

Autonomous Equations and Population Dynamics

1. Autonomous Equations

First order differential equations relate the slope of a function to the values of the function and the independent variable. This means that, writing the differential equation as $y' = f(y, t)$, we can visualize solutions by plotting bits of slope at the appropriate $y - t$ coordinate points and trying to connect them.

This is, in general, (especially the final step – connecting the slope bits in a reasonably accurate way) a massive pain. The curves that correspond to certain slopes can be complicated, which can make it hard to look at a diagram and know which initial values correspond to which outcomes. However, for a certain class of equations, called *autonomous equations*, this process is greatly simplified. Autonomous equations don't depend on t, *i.e.*,

$$y' = f(y),$$

which is nice because it means we don't have to worry about being too precise with regard to the t coordinate.

REMARK. Notice that any autonomous equation is separable.

What we need to know to study these qualitatively is which values of y make y' zero, positive, or negative. The first group are called the *equilibrium solutions* of the differential equation. They are constant solutions, and we indicate them on the $y - t$ graph by horizontal lines.

After that, we can study the positivity of $f(y)$ on the intermediate intervals, which will tell us whether the equilibrium solutions attract nearby initial conditions (in which case they are called *asymptotically stable*), repel them (*unstable*), or some combination thereof (*semi-stable*).

EXAMPLE 6.1. Consider $y' = y^2 - y - 2$. We'll start by finding the equilibrium solutions, which, in this case, are the roots of $y^2 - y - 2 = (y - 2)(y + 1)$. So the equilibrium solutions are $y = -1$ and $y = 2$. These are constant solutions, indicated on the solution space by horizontal lines. We want to understand their stability. If we plot $y^2 - y - 2$ versus y, we can see that on the interval $(-\infty, -1)$, $f(y) > 0$ while on the interval $(-1, 2)$, $f(y) < 0$. So if we have a solution with initial condition $y(t_0) = y_0 < -1$, $y' = f(y) > 0$ and $y(t)$ will increase towards -1. If $-1 < y_0 < 2$, $y' < 0$, so the solution will decrease towards -1. Since solutions starting below -1 to go -1 and so do solutions above it, we conclude $y(t) = -1$ is an asymptotically stable equilibrium. Now, we can also see that for $y_0 > 2$, $f(y) > 0$, so we see that if we start with an initial condition near but not equal to 2, the solution will move away from 2. So $y(t) = 2$ is an unstable equilibrium. □

EXAMPLE 6.2. Now let's take $y' = (y - 4)(y + 1)^2$. The equilibrium solutions are $y = -1$ and $y = 4$. To classify them, we graph $f(y) = (y - 4)(y + 1)^2$. If $y < -1$, we can see that $f(y) < 0$, so solutions starting below -1 will tend towards $-\infty$. If $-1 < y < 4$, $f(y) < 0$ again, so solutions starting in the relevant region will tend towards -1. So we have that $y = -1$ is a semi-stable solution. Finally, since for $y > 4$, $f(y) > 0$, solutions starting above 4 will asymptotically increase to ∞, and we have that $y = 4$ is unstable since no nearby solutions converge to it. □

2. Populations

Some good examples of autonomous equations come from population dynamics. The most naive population model is the so-called "population bomb" model:

$$P'(t) = rP(t)$$

with $r > 0$. This differential equation is solved by $P(t) = P_0 e^{rt}$, which indicates that the population would increase exponentially to infinity, since there's always a net growth proportional to the population. This is not very realistic at all.

A better model is the so-called "logistic equation," given by

$$\begin{aligned} P'(t) &= rP\left(1 - \frac{P}{N}\right) \\ &= rP - \frac{r}{N}P^2, \end{aligned}$$

where $N > 0$ is some constant (we'll see what it means later). Looking at this model, we see that while we still have a birth rate of rP, we also have a mortality rate proportional to P^2.

First, let's solve this equation:

$$\begin{aligned} \frac{dP}{P\left(1 - \frac{P}{N}\right)} &= r dt \\ \int \left(\frac{1}{P} + \frac{\frac{1}{N}}{1 - \frac{P}{N}}\right) dP &= \int r\,dt \\ \ln|P| - \ln\left|1 - \frac{P}{N}\right| &= rt + c \\ \frac{P}{1 - \frac{P}{N}} &= Ae^{rt} \\ P &= Ae^{rt} - \frac{1}{N}Ae^{rt}P \\ P(t) &= \frac{Ae^{rt}}{1 + \frac{A}{N}e^{rt}} \\ &= \frac{AN}{Ne^{-rt} + A} \end{aligned}$$

and if $P(0) = P_0$, we solve and get $A = \frac{P_0 N}{N - P_0}$ to yield

$$P(t) = \frac{P_0 N}{(N - P_0)\,e^{-rt} + P_0}.$$

Great...but that solution doesn't tell us a ton just by glancing at it. Let's apply the methods from earlier this lecture to see what we can get.

Looking at the logistic equation, we can see that our equilibrium solutoins ar $P = 0$ and $P = N$. Graphing $f(P) = rP\left(1 - \frac{P}{N}\right)$, we see that for $P < 0$, $f(P) < 0$, for $0 < P < N$, $f(P) > 0$, and for $P > N$, $f(P) < 0$. Thus 0 is unstable while N is asymptotically stable, and we can conclude that for any initial $P_0 > 0$,

$$\lim_{t \to \infty} P(t) = N.$$

So what is N? It's the carrying capacity of the environment. If the population exists, it will grow towards N, but the closer it gets to N the slower the population will grow. If the population somehow starts off larger than the environment can support, it will die off until it reaches that critical position. And if the population starts off at N, the births and deaths will balance out perfectly.

It's also possible to construct similar models that have unstable equilibria above 0.

EXERCISE. Show that the equilibrium $P(t) = N$ is unstable for the autonomous equation $P'(t) = rP\left(\frac{P}{N} - 1\right)$.

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