Math 251 (§10) Homework 8

Due: Wednesday, April 15, 2009

Answers to the following should be turned in no later than the end of class on the above date. *Write your name on the assignment.* This assignment is worth a total of 50 points.

QUESTION 1: Find all eigenvalues and eigenfunctions for the following boundary value problems:

i) $y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y'(l) = 0$

SOLUTION: We start by distinguishing the 3 cases giving us different general solutions:

(1) $\lambda > 0$:
Here we write $\lambda = \beta^2$. We have

$$y(x) = A\cos(\beta x) + B\sin(\beta x),$$

and so

$$y'(x) = -\beta A\sin(\beta x) + \beta B\cos(\beta x).$$

Using our first initial condition, we see that $B = 0$. The second initial condition gives that either $\beta A = 0$ or $\sin(\beta l) = 0$. Since $\beta = \sqrt{\lambda} \neq 0$, the first possibility would mean $A = 0$, but then $y(x) = 0$ and we don't want that. So we conclude that $\sin(\beta l) = 0$, and thus

$$\beta l = n\pi$$

for any integer n. Thus, for each integer n, we have an appropriate value of β,

$$\beta_n = \frac{n\pi}{l},$$

and so the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2$$

with corresponding solutions

$$y(x) = A\cos\left(\frac{n\pi x}{l}\right).$$

(2) $\lambda < 0$:
The argument here is similar for the example we did in class; we get that there are no negative eigenvalues.

(3) $\lambda = 0$:
If $\lambda = 0$, our equation is

$$y'' = 0,$$

and so

$$y(x) = Ax + B,$$

with $y'(x) = A$. Both boundary conditions say that $A = 0$, and so we're left with

$$y(x) = B.$$

Thus $\lambda_0 = 0$ is an eigenvalue with eigenfunction $y(x) = 1$.

ii) $y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(l) = 0$

SOLUTION: Again, we start by distinguishing the three cases.

(1) $\lambda > 0$:
If $\lambda = \beta^2$, the general solution is

$$y(x) = A\cos(\beta x) + B\sin(\beta x).$$

The first boundary condition tells us that $A = 0$, and so

$$y(x) = B\sin(\beta x),$$

with

$$y'(x) = \beta B\cos(\beta x).$$

Since we don't want $B = 0$, the second condition turns into $\cos(\beta l) = 0$, and so

$$\beta l = \frac{(2n+1)\pi}{2},$$

so we have, for each integer n, an appropriate value of β,

$$\beta_n = \frac{(2n+1)\pi}{2l}.$$

Thus the eigenvalues are

$$\lambda_n = \left(\frac{(2n+1)\pi}{2l}\right)^2$$

with eigenfunctions

$$y(x) = \cos\left(\frac{(2n+1)\pi x}{2l}\right).$$

(2) $\lambda < 0$:
Again, a similar argument here shows that there are no negative eigenvalues.

(3) $\lambda = 0$:
An argument similar to class shows that 0 is not an eigenvalue.

iii) $y'' + \lambda y = 0, \qquad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi)$

SOLUTION:

(1) $\lambda > 0$:
Letting $\lambda = \beta^2$, we get a general solution of

$$y(x) = A\cos(\beta x) + B\sin(\beta x),$$

and

$$y'(x) = -\beta A\sin(\beta x) + \beta B\cos(\beta x).$$

The two initial conditions then become

$$\begin{aligned} A\cos(-\beta\pi) + B\sin(-\beta\pi) &= A\cos(\beta\pi) + B\sin(\beta\pi) \\ A\cos(\beta\pi) - B\sin(\beta\pi) &= A\cos(\beta\pi) + B\sin(\beta\pi) \\ 2B\sin(\beta\pi) &= 0 \end{aligned}$$

and

$$\begin{aligned} -\beta A\sin(-\beta\pi) + \beta B\cos(-\beta\pi) &= -\beta A\sin(beta\pi) + \beta B\cos(\beta\pi) \\ A\sin(\beta\pi) + B\cos(\beta\pi) &= -A\sin(\beta\pi) + B\cos(\beta\pi) \\ 2A\sin(\beta\pi) &= 0. \end{aligned}$$

For the first equation, either $B = 0$ or $\beta = n$ for any integer n. For the second equation, either $A = 0$ or $\beta = n$. We don't want $A = B = 0$, so our conclusion is that we may take, for any integer n, $\beta_n = n$, and so our eigenvalues are $\lambda_n = n^2$ with eigenfunctions

$$y_{n,1}(x) = A\cos(nx)$$
$$y_{n,2}(x) = B\cos(nx).$$

(2) $\lambda < 0$:
This is again an argument somewhat similar to class.

(3) $\lambda = 0$:
We have $y(x) = Ax + B$. The two conditions say

$$-A\pi + B = A\pi + B$$
$$A = A.$$

So the first equation says $A = 0$, but B can still be anything at all. So $\lambda = 0$ is an eigenvalue with eigenfunction $y_0(x) = 1$.

Question 2: Reduce the following PDEs to a pair of ODEs using separation of variables.

i) $xu_{xx} + u_t = 0$

SOLUTION: We set $u(x,t) = X(x)T(t)$. Plugging this in, the PDE becomes

$$xX''(x)T(t) + T'(t)X(x) = 0.$$

Putting the second term over and separating variables, we get

$$\frac{xX''(x)}{X(x)} = -\frac{T'(t)}{T(t)} = -\lambda,$$

upon noticing that the left hand side only depends on x while the right hand side only depends on t, yet they're equal. Thus they must be equal to some constant λ. This gives rise to the two ODEs:

$$T' = \lambda T$$
$$X'' = -\frac{\lambda X}{x}.$$

ii) $x^{-1}u_t = (xu_x)_x$

SOLUTION: Proceeding as above, we have $u(x,t) = X(x)T(t)$, and so

$$x^{-1}X(x)T'(t) = xX''(x)T(t) + X'(x)T(t) = T(t)\left(xX''(x) + X'(x)\right).$$

Thus, the separation yields

$$\frac{T'(t)}{T(t)} = \frac{x^2X''(x) + xX'(x)}{X(x)} = -\lambda,$$

or

$$T' = -\lambda T$$
$$x^2X'' + xX' = -\lambda X.$$

iii) $u_{xx} + u_{tt} + xu = 0$

SOLUTION: Plugging in the form for u, we get

$$X''(x)T(t) + T''(t)X(x) + xT(t)X(x) = 0,$$

or

$$X''(x)T(t) + xX(x)T(t) = -T''(t)X(x).$$

Separating gives

$$\frac{X''(x) + xX(x)}{X(x)} = -\frac{T''(t)}{T(t)} = -\lambda,$$

or

$$T'' = \lambda T$$
$$X'' + xX = -\lambda X.$$

QUESTION 3: Find the separated solution to the following heat equation problem on a rod of length 1.

$$u_t = 100u_{xx}$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = \sin(2\pi x) - \sin(5\pi x).$$

SOLUTION: Seeing that the boundary conditions are homogeneous Dirichlet, we know that the form of the general separated solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-100n^2\pi^2 t} \sin(n\pi x),$$

as $k = 100$ and $l = 1$. Plugging in the initial condition gives

$$\sin(2\pi x) - \sin(5\pi x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x),$$

and we see that $A_2 = 1$, $A_5 = -1$, and all other $A_n = 0$. Thus the solution is

$$u(x,t) = e^{-400\pi^2 t} \sin(2\pi x) - e^{-2500\pi^2 t} \sin(5\pi x).$$

QUESTION 4: Suppose we have the heat equation $u_t = ku_{xx}$ with the following boundary conditions:

$$u(0,t) = 0$$
$$u_x(l,t) + u(l,t) = 0.$$

We know that after separation of variables, we have the following pair of ODEs:

$$T' + \lambda kT = 0$$
$$X'' + \lambda X = 0.$$

Determine the appropriate boundary conditions for $X(x)$. For which value(s) of λ do we have nontrivial solutions? *Note:* You don't have to explicitly find the positive eigenvalues, but find a condition which they have to satisfy. You should show that there are no negative or zero eigenvalues.

SOLUTION: We have the assumption that $u(x,t) = X(x)T(t)$. The two boundary conditions then become

$$X(0)T(t) = 0$$
$$X'(l)T(t) + X(l)T(t) = 0.$$

We don't want $T(t) = 0$ (since then $u(x,t) = 0$), so we get boundary conditions of $X(0) = 0$ and $X'(l) + X(l) = 0$. The argument that there are no negative or zero eigenvalues should be standard at this point, and plugging the conditions in for $\lambda = \beta^2$,, the condition we get is that $\beta = -\tan(\beta)$, so the eigenvalues would be the solutions to that equation squared.