Math 251 (§3) Homework 6

Due: Thursday, March-20-2008

Answers to the following should be turned in no later than the end of class on the above date. *Write your name on the assignment.* This assignment is worth a total of 50 points.

QUESTION 1: What are the Laplace transforms of the following?

i) $f(t) = e^{2t}\sin(t)$

Solution: We can recall from class (or see from a table of Laplace transforms) that $\mathcal{L}\{e^{at}g(t)\} = G(s-a)$, where $G(s) = \mathcal{L}\{g(t)\}$. Thus

$$\mathcal{L}\left\{e^{2t}\sin(t)\right\} = \frac{1}{(s-2)^2+1}.$$

□

ii) $f(t) = u_2(t)\left(t^2 - 2t\right)$

Solution: To take this Laplace transform, we need to write $t^2 - 2t = g(t-2)$ for some function $g(t)$.

$$\begin{aligned} g(t-2) = t^2 - 2t &= ((t-2)+2)^2 - 2((t-2)) + 2 \\ &= (t-2)^2 + 4(t-2) + 4 - 2(t-2) - 4 \\ &= (t-2)^2 + 2(t-2) \end{aligned}$$

Hence we have $g(t) = t^2 + 2t$ and the Laplace transform is

$$\mathcal{L}\left\{u_2(t)(t^2-2t)\right\} = e^{-2s}\mathcal{L}\left\{t^2+2t\right\} = e^{-2s}\left(\frac{2}{s^3} - \frac{2}{s^2}\right).$$

□

iii) $f(t) = 2t^3 e^{-\frac{t}{2}}$

Solution: This is similar to part i). We have

$$\mathcal{L}\left\{2t^3 e^{-\frac{t}{2}}\right\} = \frac{12}{\left(s+\frac{1}{2}\right)^4}.$$

□

iv) $f(t) = u_\pi(t)\cos(3t)$

Solution:

$$\begin{aligned} g(t-\pi) = \cos(3t) &= \cos(3((t-\pi)+\pi) \\ &= \cos(3(t-\pi))\cos(3\pi) - \sin(3(t-\pi))\sin(3\pi) \\ &= -\cos(3(t-\pi)) \end{aligned}$$

So $g(t) = -\cos(3t)$ and

$$\mathcal{L}\left\{u_\pi(t)\cos(3t)\right\} = e^{-\pi s}\mathcal{L}\left\{-\cos(3t)\right\} = -e^{-\pi s}\frac{s}{s^2+9}.$$

□

v) $f(t) = u_{\frac{\pi}{2}}(t)e^t\sin(t)$

Solution:

$$\begin{aligned} g\left(t-\frac{\pi}{2}\right) = e^t\sin(t) &= e^{\left(t-\frac{\pi}{2}\right)+\frac{\pi}{2}}\sin\left(\left(t-\frac{\pi}{2}\right)+\frac{\pi}{2}\right) \\ &= e^{\frac{\pi}{2}}e^{t-\frac{\pi}{2}}\left(\sin\left(t-\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \cos\left(t-\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)\right) \\ &= e^{\frac{\pi}{2}}e^{t-\frac{\pi}{2}}\cos\left(t-\frac{\pi}{2}\right) \end{aligned}$$

So

$$g(t) = e^{\frac{\pi}{2}} e^t \cos(t),$$

and

$$\mathcal{L}\left\{u_{\frac{\pi}{2}}(t)e^t \sin(t)\right\} = e^{-\frac{\pi}{2}s}\mathcal{L}\left\{e^{\frac{\pi}{2}} e^t \cos(t)\right\} = e^{-\frac{\pi}{2}s} e^{\frac{\pi}{2}} \frac{s-1}{(s-1)^2+1}.$$

□

QUESTION 2: What are the inverse Laplace transforms of the following?

i) $F(s) = e^{-2s}\dfrac{2}{s^3}$

Solution: The only thing to keep in mind here is that, as the e^{-2s} will get turned into $u_2(t)$ by the inverse transform, we'll need to shift the inverse transform of $\frac{2}{s^3}$ by 2. So we end up with

$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{2}{s^3}\right\} = u_2(t)(t-2)^2.$$

□

ii) $F(s) = \dfrac{4}{s^2 - 6s + 10}$

Solution: The denominator doesn't factor, so the first thing we need to do is to complete the square in the denominator. This gives

$$F(s) = \frac{4}{(s-3)^2+1},$$

and our inverse transform is

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2-6s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{(s-3)^2+1}\right\} = 4e^{3t}\sin(t).$$

□

iii) $F(s) = \dfrac{2s+3}{s^2+1}$

Solution: All we have to do here is split up this fraction.

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} + \frac{3}{s^2+1}\right\} = 2\cos(t) + 3\sin(t).$$

□

iv) $F(s) = \dfrac{3s+5}{(s+1)^2+16}$

Solution:

$$\begin{aligned} F(s) &= \frac{3((s+1)-1)+5}{(s+1)^2+16} \\ &= \frac{3(s+1)+2}{(s+1)^2+16} \\ &= \frac{3(s+1)}{(s+1)^2+16} + \frac{2}{(s+1)^2+16} \\ &= \frac{3(s+1)}{(s+1)^2+16} + \frac{1}{2}\frac{4}{(s+1)^2+16} \end{aligned}$$

Thus the inverse transform is

$$\mathcal{L}^{-1}\left\{\frac{3s+5}{(s+1)^2+16}\right\} = 3e^{-t}\cos(4t) + \frac{1}{2}e^{-t}\sin(4t).$$

□

v) $F(s) = \dfrac{4s^3 - 3s^2 + 6s + 10}{s^2\,(s^2 - 2s + 5)}$

Solution: We need to first do partial fractions to split up this term. Doing so gives

$$F(s) = \frac{2}{s} + \frac{2}{s^2} + \frac{2s-1}{s^2 - 2s + 5}.$$

The first two terms are perfect for inverse transforming, but we need to do something with the third. Completing the square on that denominator gives

$$s^2 - 2s + 5 = (s-1)^2 + 4,$$

and doing work analogous to that in the previous example gives

$$F(s) = \frac{2}{s} + \frac{2}{s^2} + 2\frac{s-1}{(s-1)^2+4} + \frac{1}{2}\frac{2}{(s-1)^2+4}.$$

The inverse transform is

$$f(t) = 2 + 2t + 2e^t\cos(2t) + \frac{1}{2}e^t\sin(2t).$$

□

vi) $F(s) = \dfrac{5s^2 - 16s + 6}{s(s-2)(s-3)}$

Solution: Doing partial fractions gives

$$F(s) = \frac{1}{s} + \frac{3}{s-2} + \frac{1}{s-3}.$$

So the inverse transform is

$$f(t) = 1 + 3e^{2t} + e^{3t}.$$

□

Question 3: Solve the following initial value problems using the method of Laplace transforms.

i) $y'' + 8y' + 52y = 13u_2(t) \qquad y(0) = 0 \quad y'(0) = 1$

Solution: After transforming, plugging in the initial conditions, and solving for $Y(s)$, we obtain

$$Y(s) = \frac{13e^{-2s}}{s(s^2+8s+52)} + \frac{1}{s^2+8s+52}.$$

We need to partial fraction the first term and complete the square on $s^2 + 8s + 52$. Doing this gives

$$Y(s) = e^{-2s}\left(\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{(s+4)^2+36} - \frac{2}{(s-4)^2+36}\right) + \frac{1}{(s+4)^2+36}.$$

Next, we need to fix a couple of numerators; doing this gives

$$Y(s) = e^{-2s}\left(\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s+4}{(s+4)^2+36} - \frac{1}{(s+4)^2+36}\right) + \frac{1}{(s+4)^2+36}$$

and the solution is

$$y(t) = u_2(t)\left(\frac{1}{4} - \frac{1}{4}e^{-4(t-2)}\cos(6(t-2)) - \frac{1}{6}e^{-4(t-2)}\sin(6(t-2))\right) + \frac{1}{6}e^{-4t}\sin(6t).$$

□

ii) $y'' + 9y = \begin{cases} 3 & t < 1 \\ 0 & 1 \le t \end{cases} \qquad y(0) = -1 \quad y'(0) = 0$

Solution: The first task is to rewrite the nonhomogeneous term using step functions. Doing this gives us a differential equation of

$$y'' + 9y = 3 - 3u_1(t).$$

Transforming and solving for $Y(s)$ gives

$$\begin{aligned} Y(s) &= \frac{3}{s(s^2+9)} - \frac{3e^{-s}}{s(s^2+9)} - \frac{s}{s^2+9} \\ &= \frac{3-s^2}{s(s^2+9)} - e^{-s}\frac{3}{s(s^2+9)}. \end{aligned}$$

After doing partial fractions on both terms, we have

$$Y(s) = \frac{1}{3}\frac{1}{s} - \frac{4}{3}\frac{s}{s^2+9} - e^{-s}\left(\frac{1}{3}\frac{1}{s} - \frac{1}{3}\frac{s}{s^2+9}\right)$$

and the solution is

$$y(t) = \frac{1}{3} - \frac{4}{3}\cos(3t) - u_1(t)\left(\frac{1}{3} - \frac{1}{3}\cos(3(t-1))\right).$$

□

iii) $y'' + 4y = 1 - 2t \qquad y(0) = 1 \quad y'(0) = 2$

Solution: Transforming and solving for $Y(s)$ yields

$$\begin{aligned} Y(s) &= \frac{1}{s(s^2+4)} - 2\frac{s^2(s^2+4)}{+}\frac{s+2}{s^2+4} \\ &= \frac{s^3+2s^2+s-2}{s^2(s^2+4)}. \end{aligned}$$

After doing partial fractions, we get

$$Y(s) = \frac{1}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \frac{3}{4}\frac{s}{s^2+4} + \frac{5}{2}\frac{1}{s^2+4},$$

and the solution is

$$y(t) = \frac{1}{4} - \frac{1}{2}t + \frac{3}{4}\cos(2t) + \frac{5}{4}\sin(2t).$$

□