

# Math 251 (§3) Homework 4 Solutions

Due: Wednesday, 13-Feb-07

Answers to the following should be turned in no later than the end of class on the above date. *Write your name on your assignment.* This assignment is worth 50 points.

QUESTION 1: Are the following pairs of functions linearly independent? Justify your answer.

i)  $\phi(x) = e^{\alpha x}$        $\psi(x) = xe^{\alpha x}$  where  $\alpha \neq 0$  is real.

**Solution:** The Wronskian of the two functions is

$$W = \begin{vmatrix} e^{\alpha x} & xe^{\alpha x} \\ \alpha e^{\alpha x} & e^{\alpha x} + xe^{\alpha x} \end{vmatrix} = e^{2\alpha x} \neq 0,$$

so we conclude that the functions are linearly independent.  $\square$

ii)  $\phi(x) = e^x \sin(x)$        $\psi(x) = e^x \cos(x)$

**Solution:** The Wronskian of the two functions is

$$W = \begin{vmatrix} e^x \sin(x) & e^x \cos(x) \\ e^x \cos(x) + e^x \sin(x) & e^x \cos(x) - e^x \sin(x) \end{vmatrix} = -e^{2x} \neq 0,$$

so we conclude that the functions are linearly independent.  $\square$

iii)  $\phi(x) = \ln x$        $\psi(x) = \ln x^2$

**Solution:** We can notice that  $\ln x^2 = 2 \ln x$ , so that  $-2\phi(x) + \psi(x) = 0$ . Thus the functions are linearly dependent.  $\square$

iv)  $\phi(x) = \frac{1}{1+t}$        $\psi(x) = \frac{1+t}{1-t^2}$

**Solution:** The Wronskian of the two functions is

$$W = \begin{vmatrix} \frac{1}{1+t} & \frac{1+t}{1-t^2} \\ -\frac{1}{(1+t)^2} & \frac{1-t^2-2t(1+t)}{(1-t^2)^2} \end{vmatrix} \neq 0,$$

so we conclude the functions are linearly independent.  $\square$

v)  $\phi(x) = \sin(2x)$        $\psi(x) = \cos(2x)$

**Solution:** The Wronskian of the two functions is

$$W = \begin{vmatrix} \sin(2x) & \cos(2x) \\ 2 \cos(2x) & -2 \sin(2x) \end{vmatrix} = -2 \neq 0,$$

so we conclude the functions are linearly independent.  $\square$

QUESTION 2: Find the particular solution to the following initial value problems.

i)  $y'' + 10y' + 25y = 0$        $y(0) = 2, y'(0) = -1$

**Solution:** The characteristic equation is

$$r^2 + 10r + 25 = 0,$$

which has a repeated root  $r = -5$ . Thus the general solution is

$$y(t) = c_1 e^{-5t} + c_2 t e^{-5t}.$$

The initial conditions give that  $c_1 = 2$  and  $c_2 = 9$  so that the particular solution is

$$y(t) = 2e^{-5t} + 9te^{-5t}.$$

$\square$

ii)  $y'' - 8y' + 16y = 0$        $y(0) = 1, y'(0) = 0$

**Solution:** The characteristic equation is

$$r^2 - 8r + 16 = 0,$$

which has a repeated root  $r = 4$ . Thus the general solution is

$$y(t) = c_1 e^{4t} + c_2 t e^{4t}.$$

The initial conditions give that  $c_1 = 1$  and  $c_2 = -4$  so that the particular solution is

$$y(t) = e^{4t} - 4t e^{4t}.$$

□

iii)  $2y'' + 8y' + 6y = 0 \quad y(0) = 0, y'(0) = 1$

**Solution:** The characteristic equation is

$$2r^2 + 8r + 6 = 0,$$

which has a roots  $r_1 = -1$  and  $r_2 = -3$ . Thus the general solution is

$$y(t) = c_1 e^{-t} + c_2 e^{-3t}.$$

The initial conditions give that  $c_1 = \frac{1}{2}$  and  $c_2 = -\frac{1}{2}$  so that the particular solution is

$$y(t) = \frac{1}{2} e^{-t} - \frac{1}{2} e^{-3t}.$$

□

iv)  $y'' - 6y' + 18y = 0 \quad y(0) = 0, y'(0) = 1$  The characteristic equation is

$$r^2 - 6r + 18 = 0,$$

which has complex roots  $r_{1,2} = 3 \pm 3i$ . The general solution is

$$y(t) = c_1 e^{3t} \cos(3t) + c_2 e^{3t} \sin(3t),$$

and the initial conditions give  $c_1 = 0$  and  $c_2 = \frac{1}{3}$ . The particular solution is

$$y(t) = \frac{1}{3} e^{3t} \sin(3t).$$

v)  $y'' + 4y' - 5y = 0 \quad y(0) = 3, y'(0) = 2$

**Solution:** The characteristic equation is

$$r^2 + 4r - 5 = 0,$$

which has roots  $r_1 = -5$  and  $r_2 = 1$ . Thus the general solution is

$$y(t) = c_1 e^{-5t} + c_2 e^t.$$

The initial conditions give that  $c_1 = \frac{1}{6}$  and  $c_2 = \frac{17}{6}$  so that the particular solution is

$$y(t) = \frac{1}{6} e^{-5t} + \frac{17}{6} e^t.$$

□

vi)  $y'' + 12y' + 40y = 0 \quad y\left(\frac{\pi}{2}\right) = 1, y'\left(\frac{\pi}{2}\right) = 2$

**Solution:** The characteristic equation is

$$r^2 + 12r + 40 = 0,$$

which has complex roots  $r_{1,2} = -6 \pm 2i$ . Thus the general solution is

$$y(t) = c_1 e^{-6t} \cos(2t) + c_2 e^{-6t} \sin(2t).$$

The initial conditions give that  $c_1 = -1$  and  $c_2 = -8$  so that the particular solution is

$$y(t) = -1e^{6t} \cos(2t) - 8e^{-6t} \sin(2t).$$

□

vii)  $y'' + 7y' + 12y = 0 \quad y(2) = 3, y'(2) = 1$

**Solution:** The characteristic equation is

$$r^2 + 7r + 12 = 0,$$

which has roots  $r_1 = -4$  and  $r_2 = -3$ . Thus the general solution is

$$y(t) = c_1 e^{-4t} + c_2 e^{-3t}.$$

The initial conditions give that  $c_1 = -10e^8$  and  $c_2 = 13e^6$  so that the particular solution is

$$y(t) = -10e^{8-4t} + 13te^{6-3t}.$$

□

QUESTION 3: Find the general solution to the following equations.

i)  $(x + 1)^2 y'' - 2(x + 1)y' + 2y = 0 \quad y_1(x) = x + 1$

**Solution:** This is a standard enough reduction of order problem. Set  $y_2(x) = v(x)y_1(x) = v(x)(x+1)$ . Differentiating, plugging in, and simplifying, we obtain

$$(x + 1)^3 v'' = 0,$$

so  $v''(x) = 0$  and  $v(x) = cx + k$ . We choose  $c = 1$  and  $k = 0$ , and then  $y_2(x) = x(x + 1)$  so that the general solution is

$$y(x) = c_1(x + 1) + c_2x(x + 1).$$

□

ii)  $xy'' + 2y' + xy = 0 \quad y_1(x) = x^{-1} \sin(x)$

**Solution:** We guess  $y_2(x) = v(x)y_1(x) = v(x)\frac{\sin(x)}{x}$ . Differentiating, plugging in, and simplifying gives

$$\sin(x)v'' + 2\cos(x)v' = 0,$$

so upon substituting  $w = v'$ , we end up with the first order linear equation

$$w' + 2\cot(x)w = 0.$$

This is solved by

$$w(x) = \frac{c}{\sin^2(x)}$$

so that

$$v(x) = c \cot(x) + k.$$

Choosing  $c = 1$  and  $k = 0$ , we have

$$y_2(x) = v(x)y_1(x) = x^{-1} \cos(x).$$

Thus the general solution is

$$y(x) = c_1 \frac{\sin(x)}{x} + c_2 \frac{\cos(x)}{x}.$$

□