Math 251 (§3) Homework 3 Solutions

Due: Wednesday, 6-Feb-07

Answers to the following should be turned in no later than the end of class on the above date. *Write your name on your assignment.* This assignment is worth 50 points.

QUESTION 1: Show that on the interval $[0, \pi]$, the functions $y_1(x) = 1$ and $y_2(x) = \cos(x)$ both satisfy the initial value problem

$$\frac{dy}{dx} + \sqrt{1 - y^2} = 0 \qquad y(0) = 1.$$

Explain why this doesn't contradict the Theorem of Existence and Uniqueness.

Solution: We show that they both solve the initial value problem by plugging in. The theorem doesn't apply to this initial value problem, since if we write $y(x) = -\sqrt{1 - y^2} = f(x, y)$, we can see that there is no y-interval around our initial $y_0 = 1$ where f is continuous, since it's not defined for any $y > 1$. □

QUESTION 2: Find the particular solution to the following initial value problems:

i) $-ye^t = (e^t + (y + 1)e^y) \qquad y(0) = 3.$

Solution: This is an exact equation, with $M(t, y) = ye^t$ and $N(t, y) = e^t + (y + 1)e^y$. M will be easier to integrate than N, so we set

$$\Psi(x, y) = \int M(t, y)\, dt = \int ye^t\, dt = ye^t + h(y).$$

Now we know that $\Psi_y(x, y) = N$, and we'll use this to figure out what h is.

$$\Psi_y = e^t + h'(y) = e^t + (y + 1)e^y = N(t, y),$$

so we conclude $h'(y) = (y + 1)e^y$ and $h(y) = ye^y$. So $\Psi(x, y) = ye^t + ye^y$ and our general solution is then

$$ye^t + ye^y = c.$$

The initial condition gives

$$3 + 3e^3 = c,$$

so our final implicit solution is

$$ye^t + ye^y = 3 + 3e^3.$$

□

ii) $2 - \tan(y) - t\sec^2(y)y' = 0 \qquad y(2) = \pi.$

Solution: Again, this is exact. $M(t, y) = 2 - \tan(y)$ and $N(t, y) = -t\sec^2(y)$. We set

$$\Psi(t, y) = \int N(t, y)\, dy = \int -t\sec^2(y)\, dy = -t\tan(y) + h(t).$$

Now we have

$$\Psi_t(t, y) = -\tan(y) + h'(t) = 2 - \tan(y) = M(t, y),$$

so $h'(t) = 2$ and $h(t) = 2t$. So

$$\Psi(t, y) = 2t - t\tan(y) = c,$$

and the initial conditions tell us that $c = 4$. The particular solution is then

$$2t - t\tan(y) = 4.$$

□

iii) $2y'' + 8y' + 6y = 0 \qquad y(0) = 1 \quad y'(0) = -3.$

Solution: The characteristic equation is

$$2r^2 + 8r + 6 = 0,$$

and this has distinct real roots $r_1 = -1$ and $r_2 = -3$. The general solution is then

$$y(t) = c_1 e^{-t} + c_2 e^{-3t},$$

and the initial conditions give $c_1 = 0$ and $c_2 = 1$, so the particular solution is

$$y(t) = e^{-3t}.$$

□

iv) $y'' - 7y' + 8y = 0 \qquad y(0) = 2 \quad y'(0) = 0.$

Solution: The characteristic equation is

$$r^2 - 7r + 8 = 0,$$

which has roots

$$r_{1,2} = \frac{7 \pm \sqrt{17}}{2}.$$

The general solution is

$$y(t) = c_1 e^{\frac{7+\sqrt{17}}{2}t} + c_2 e^{\frac{7-\sqrt{17}}{2}t}.$$

The initial conditions give $c_1 = \frac{\sqrt{17}-7}{\sqrt{17}}$ and $c_2 = \frac{7+\sqrt{17}}{\sqrt{17}}$. The particular solution is then

$$y(t) = \frac{\sqrt{17}-7}{\sqrt{17}} e^{\frac{7+\sqrt{17}}{2}t} + \frac{\sqrt{17}+7}{\sqrt{17}} e^{\frac{7-\sqrt{17}}{2}t}.$$

□

v) $y'' + 12y' + 20y = 0 \qquad y(1) = 2 \quad y'(1) = -1.$

Solution: The characteristic equation is

$$r^2 + 12r + 20 = 0,$$

which has roots $r_1 = -10$ and $r_2 = -2$. The general solution is

$$y(t) = c_1 e^{-10t} + c_2 e^{-2t},$$

and the initial conditions give $c_1 = \frac{3}{8}e^{10}$ and $c_2 = \frac{13}{8}e^2$, so the particular solution is

$$y(t) = \frac{3}{8}e^{10-10t} + \frac{13}{8}e^{2-2t}.$$

□

vi) $y'' - 16y = 0 \qquad y(0) = 5 \quad y'(0) = 6.$

Solution: The characteristic equation is

$$r^2 - 16 = 0,$$

which has roots $r_{1,2} = \pm 4$. The general solution is

$$y(t) = c_1 e^{4t} + c_2 e^{-4t},$$

and the initial conditions give $c_1 = \frac{13}{4}$ and $c_2 = \frac{7}{4}$ so that the particular solution is

$$y(t) = \frac{13}{4}e^{4t} + \frac{7}{4}e^{-4t}.$$

□

QUESTION 3: The point of this problem is to show that the principle of superposition does not in general hold for nonlinear equations.

i) Show that $y_1 = 1$ and $y_2 = \sqrt{x}$ are solutions to $yy'' + (y')^2 = 0$ but that their sum $y = y_1 + y_2$ is not.

Solution: Just plug everything in and see which functions work and which ones don't. □

ii) Show that $y = x^3$ is a solution of $yy'' = 6x^4$. For which value(s) of c is $y = cx^3$ a solution?

Solution: Plugging in shows that this only work for $c = \pm 1$. □