# Math 251 (§3) Homework 2

## Due: Wednesday, 30-Jan-07

Answers to the following should be turned in no later than the end of class on the above date. *Write your name on your assignment.* This assignment is worth 50 points.

QUESTION 1: You are sitting in a hot tub in the snow. Snow enters the tub at a rate of 1 liter per minute and water flows out of the tub at the same rate. Initially, the tub holds 500 liters of water and has a chlorine concentration of 10 milligrams per liter. The filter adds 2 milligrams per minute of chlorine. If we assume the hot tub is well mixed:

i) Write down and solve the initial value problem for the amount of chlorine in the hot tub.

**Solution:** If $P(t)$ is the amount of chlorine in the pool after $t$ minutes:

$$P' = \frac{\text{mg added}}{\text{minute}} - \frac{\text{mg removed}}{\text{minute}}$$
$$P' = 2 - \frac{P}{500}.$$

For the initial condition, we need to be slightly careful. We're given the initial *concentration* of chlorine, but $P(t)$ is the amount of chlorine. So we need to multiply concentration by volume and we get $P(0) = 5000$.
This is a fairly standard integrating factor problem. The solution is

$$P(t) = 1000 + 4000e^{-\frac{t}{500}}.$$

□

ii) If the filter can only add 60 milligrams of chlorine, how much chlorine will be present in the tub when it runs out?

**Solution:** The filter can only add 60 mg of chlorine and it adds 2 mg per minute. Thus it will run out after 30 minutes, so we want to know $P(30)$. Plugging in:

$$P(30) == 1000 + 4000e^{-\frac{3}{50}} \approx 4767\text{mg}.$$

□

QUESTION 2: A 100 liter well-mixed tank initially contains 100 liters of water contaminated by 20 grams of blue food coloring. Suppose that food coloring is added at a constant rate of $a$ grams per minute while water is drained out at a rate of 1 liter per minute. What value of $a$ should be chosen so that the concentration of coloring is 3 grams per liter when 20 liters remains?

**Solution:** As in the earlier mixing problem, if $P(t)$ is the amount of food coloring in the tank after $t$ minutes, $P'(t)$ will be given by the difference between the adding rate and the leaving rate:

$$P'(t) = a - \frac{P}{100 - t}$$
$$P(0) = 20.$$

Again, this is a straightforward integrating factor problem, and the solution is

$$P(t) = (100 - t)\left(-a\ln(100 - t) + \frac{1}{5} + a\ln(100)\right).$$

Now we want to know which value of $a$ will give us a concentration of 3 grams per liter when there are 20 liters remaining. There will be

20 liters remaining at $t = 80$ and so the condition we need to use is $P(80) = 60$. So:

$$\begin{aligned} 60 &= 20(-a\ln(20) + \frac{1}{5} + a\ln(100)) \\ 3 &= a(\ln(100) - \ln(20)) + \frac{1}{5} \\ \frac{14}{5} &= a\ln(5) \\ a &= \frac{14}{5\ln(5)}. \end{aligned}$$

□

QUESTION 3: For the following DEs, find and classify their critical points and sketch the phase space. Then note the behavior of the solution with the given initial value.

i) $y' = y^3 - 2y^2 - 5y + 6, \quad y(0) = 1$

**Solution:** With $y' = f(y)$, we have $f(y) = y^3 - 2y^2 - 5y + 6$. The roots of this cubic are $y = 1, 3, -2$. So these are the critical points. Graphing $f$, we see that it is positive on the intervals $(-2, 1)$ and $(3, \infty)$, while it is negative on $(-\infty, -2)$ and $(1, 3)$. We conclude that $y = -2$ is an unstable equilibrium, $y = 1$ is an asymptotically stable equilibrium, and $y = 3$ is again unstable. □

ii) $y' = (y + 1)\sin(y), \quad y(0) = \pi$.

**Solution:** The roots of $f(y) = (y + 1)\sin(y)$ are $y = -1$ and $y = n\pi$ for any integer $n$. Graphing $f(y)$, we can conclude that $y = -1$ is asymptotically stable, $y = 2n\pi$ is unstable if $n \geq 0$ and stable if $n < 0$, and $y = (2n+1)\pi$ is asymptotically stable if $n \geq 0$ and unstable if $n < 0$. □

QUESTION 4: For each of the following differential equations, state whether they are linear or nonlinear. Then, depending on your answer, conclude what you can from the Fundamental Theorem of Existence and Uniqueness.

i) $y^3 y' = e^t \tan(yt) \qquad y(1) = 17\pi$

**Solution:** This is definitely nonlinear due to both the $y^3 y'$ and the $\tan(yt)$. So we need to use the more general version of the Existence\Uniqueness Theorem, and we write:

$$y' = \frac{e^t \tan(yt)}{y^3} = f(t, y).$$

We want to find the discontinuities of $f$ and $f_y$. These occur at

$$y = 0, yt = \frac{(2n + 1)\pi}{2}.$$

Our initial condition is $y(1) = 17\pi$, so only the second of these possibilities will matter, and these are an infinite family of hyperbolas. The two neighboring $(1, 17\pi)$ are

$$yt = \frac{33\pi}{2}, \frac{35\pi}{2}$$

and so the largest $t$-interval surrounding $(1, 17\pi)$ with no discontinuities is

$$\frac{33}{34} < t < \frac{35}{34}$$

and we can only conclude that this IVP has a unique solution defined somewhere inside of this interval. □

ii) $y' + \frac{1}{x^2-1} = \ln(\sin x) \qquad y\left(\frac{1}{10}\right) = 4$

**Solution:** This is a linear ODE so we'll use the special case. The coefficient functions have discontinuities at $x = \pm 1$ or $(2n+1)\pi \leq x \leq 2n\pi$. Our initial $x_0 = \frac{1}{10}$, and so the largest good $x$-interval containing $x_0$ is $(0, 1)$. Since we're using the linear version of the theorem our conclusion is that there exists a solution that will be unique on $(0, 1)$. □

iii) $\sin(2x)y'' + \frac{1}{2x-1}y' + x^3 y = 2 \qquad y\left(\frac{1}{4}\right) = y'\left(\frac{1}{4}\right) = 3$

**Solution:** Again, this is a linear ODE. Putting it into the proper form for the theorem, we have

$$y'' + \frac{1}{(2x-1)\sin(2x)}y' + \frac{x^3}{\sin(2x)} = \frac{2}{\sin(2x)}.$$

The discontinuities of the coefficient functions are at $x = \frac{1}{2}$ and $x = 2n\pi$ for any integer $n$. Our initial $x_0 = \frac{1}{4}$, so we conclude that there is a unique solution with interval of validity $\left(0, \frac{1}{2}\right)$. □