

Symbolic dynamics:
entropy = dimension = complexity

Stephen G. Simpson
Department of Mathematics
Pennsylvania State University
<http://www.math.psu.edu/simpson>
simpson@math.psu.edu

First draft: March 17, 2010
This draft: February 29, 2012

Abstract

Let d be a positive integer. Let G be the additive monoid \mathbb{N}^d or the additive group \mathbb{Z}^d . Let A be a finite set of symbols. The shift action of G on A^G is given by $S^g(x)(h) = x(g+h)$ for all $g, h \in G$ and all $x \in A^G$. A G -subshift is defined to be a nonempty closed set $X \subseteq A^G$ such that $S^g(x) \in X$ for all $g \in G$ and all $x \in X$. Given a G -subshift X , the topological entropy $\text{ent}(X)$ is defined as usual [31]. The standard metric on A^G is defined by $\rho(x, y) = 2^{-|F_n|}$ where n is as large as possible such that $x|_{F_n} = y|_{F_n}$. Here $F_n = \{0, 1, \dots, n\}^d$ if $G = \mathbb{N}^d$, and $F_n = \{-n, \dots, -1, 0, 1, \dots, n\}^d$ if $G = \mathbb{Z}^d$. For any $X \subseteq A^G$ the Hausdorff dimension $\dim(X)$ and the effective Hausdorff dimension $\text{effdim}(X)$ are defined as usual [14, 26, 27] with respect to the standard metric. It is well known that $\text{effdim}(X) = \sup_{x \in X} \liminf_n \text{K}(x|_{F_n})/|F_n|$ where K denotes Kolmogorov complexity [9]. If X is a G -subshift, we prove that $\text{ent}(X) = \dim(X) = \text{effdim}(X)$, and $\text{ent}(X) \geq \limsup_n \text{K}(x|_{F_n})/|F_n|$ for all $x \in X$, and $\text{ent}(X) = \lim_n \text{K}(x|_{F_n})/|F_n|$ for some $x \in X$.

Contents

Abstract	1
Contents	2
1 Introduction	2
2 How this paper came about	3
3 Background	4
3.1 Topological entropy	5
3.2 Symbolic dynamics	7
3.3 Hausdorff dimension	8
3.4 Kolmogorov complexity	8
3.5 Effective Hausdorff dimension	9
3.6 Measure-theoretic entropy	10
4 Entropy = dimension	12
5 Dimension = complexity	14
References	16

1 Introduction

The purpose of this paper is to elucidate a close relationship among three disparate concepts which are known to play a large role in three diverse branches of contemporary mathematics. The concepts are:

entropy, Hausdorff dimension, Kolmogorov complexity.

Although it is widely felt that these concepts are somehow related, it seems to us that the full depth of the relationship has been insufficiently appreciated. Below we attempt to rectify this situation by proving that, in an important special case, all three concepts coincide.

Here is a brief overview of the above-mentioned concepts.

1. *Hausdorff dimension* is a basic concept in metric geometry. See for instance the original paper by Hausdorff [14] and the classic treatise by C. A. Rogers [29]. To any set X in a metric space one assigns a nonnegative real number $\dim(X)$ = the Hausdorff dimension of X . In the case of smooth sets such as algebraic curves and surfaces, the Hausdorff dimension is a nonnegative integer and coincides with other familiar notions of dimension from algebra, differential geometry, etc. For example, the Hausdorff dimension of a smooth surface in n -dimensional space is 2. On the other hand, Hausdorff dimension applies also to non-smooth sets with nonintegral dimension, e.g., fractals and Julia sets [10].

2. *Kolmogorov complexity* plays an important role in information theory [7, 32], theoretical computer science [18, 37], and recursion/computability theory [9, 22]. To each finite mathematical object τ one assigns a nonnegative integer $K(\tau)$ = the complexity of τ . Roughly speaking, $K(\tau)$ is the length in bits of the shortest computer program which describes τ . In this sense $K(\tau)$ measures the “amount of information” which is inherent in τ .
3. *Entropy* is an important concept in dynamical systems theory [8]. Classically, a dynamical system consists of a set X together with a mapping $T : X \rightarrow X$ and one studies the long-term behavior of the orbits $\langle T^n(x) \mid n = 0, 1, 2, \dots \rangle$ for each $x \in X$. More generally, one considers an action T of a group or semigroup G on a set X , and then the orbit of $x \in X$ is $\langle T^g(x) \mid g \in G \rangle$. The entropy of the system X, T is a nonnegative real number which has a rather complicated definition but is intended to quantify the “exponential growth rate” of the system.

An especially useful class of dynamical systems are the symbolic systems, a.k.a., subshifts [19, 33]. Given a finite set of symbols A , one defines the shift action of G on A^G as usual. A subshift is then defined to be a closed, shift-invariant subset of A^G . These symbolic systems play a large role in general dynamical systems theory, because for any dynamical system X, T one can consider partitions $\pi : X \rightarrow A$ and then the behavior of an orbit $\langle T^g(x) \mid g \in G \rangle$ is reflected by the behavior of its “symbolic trace,” $\langle \pi(T^g(x)) \mid g \in G \rangle$, which is a point in A^G .

Our main results in this paper are Theorems 4.2 and 5.3 below. They say the following. Let d be a positive integer, let G be the additive monoid \mathbb{N}^d or the additive group \mathbb{Z}^d , let A be a finite set of symbols, and let $X \subseteq A^G$ be a subshift. Then, the entropy of X is equal to the Hausdorff dimension of X with respect to the standard metric on A^G . Moreover, the entropy of X has a sharp characterization in terms of the Kolmogorov complexity of the finite configurations which occur in the orbits of X .

In connection with our characterization of entropy in terms of Kolmogorov complexity, it is interesting to note that both of these concepts originated with A. N. Kolmogorov, but in different contexts [35, 17].

2 How this paper came about

This paper is an outcome of my reading and collaboration over the past several years. Here are some personal comments on that process.

It began with my study of Bowen’s alternative definition of topological entropy [3, pages 125–126]. Obviously Bowen’s definition resembles the standard definition of Hausdorff dimension in a metric space, and this led me to consider the following question:

Given a subshift X , what is the precise relationship between the topological entropy of X and the Hausdorff dimension of X ?

Specifically, let A be a finite set of symbols. From [3, Proposition 1] it was clear to me that the topological entropy of a one-sided subshift $X \subseteq A^{\mathbb{N}}$ is equal to the Hausdorff dimension of X with respect to the standard metric. And eventually I learned that this result appears explicitly in Furstenberg 1967 [11, Proposition III.1]. But what about other kinds of subshifts on A ? For instance, what about the two-sided case, i.e., subshifts in $A^{\mathbb{Z}}$ or $A^{\mathbb{Z}^d}$ or more generally A^G where G is a countable amenable group [38]? And what about the general one-sided case, i.e., subshifts in $A^{\mathbb{N}^d}$ or more generally A^G where G is countable amenable semigroup, whatever that may mean?

During February, March and April of 2010 I discussed these issues with several colleagues: John Clemens, Vaughn Climenhaga [6, Example 4.1], Manfred Denker [8], Michael Hochman [15], Anatole Katok [16], Daniel Mauldin, Yakov Pesin [25], Jan Reimann [26], Alexander Shen [37], Daniel Thompson, Jean-Paul Thouvenot [16]. All of these discussions were extremely helpful. In particular, Hochman and Mauldin provided several ideas which play an essential role in this paper.

3 Background

In this section we present some background material concerning symbolic dynamics, entropy, Hausdorff dimension, and Kolmogorov complexity. All of the concepts and results in this section are well known.

We write

$$\mathbb{N} = \{0, 1, 2, \dots\} = \{\text{the nonnegative integers}\}$$

and

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\} = \{\text{the integers}\}.$$

Throughout this paper, let G be the additive monoid \mathbb{N}^d or the additive group \mathbb{Z}^d where d is a fixed positive integer. An *action* of G on a set X is a mapping $T : G \times X \rightarrow X$ such that $T^e(x) = x$ and $T^g(T^h(x)) = T^{g+h}(x)$ for all $g, h \in G$ and all $x \in X$. Here e is the identity element of G . It is useful to write G in a specific¹ way as the union of a sequence of finite sets, namely $G = \bigcup_{n=0}^{\infty} F_n$ where $F_n = \{0, 1, \dots, n\}^d$ if $G = \mathbb{N}^d$, and $F_n = \{-n, \dots, -1, 0, 1, \dots, n\}^d$ if $G = \mathbb{Z}^d$. In particular we have $F_0 = \{0\}^d = \{e\}$. We also write $F_{-1} = \emptyset =$ the empty set. For any finite set F we write $|F| =$ the cardinality of F . For any function Φ we write $\text{dom}(\Phi) =$ the domain of Φ , and $\text{rng}(\Phi) =$ the range of Φ , and

$$\Phi : \subseteq X \rightarrow Y$$

meaning that Φ is a function with $\text{dom}(\Phi) \subseteq X$ and $\text{rng}(\Phi) \subseteq Y$. Apart from this, all of our set-theoretic notation is standard.

¹In particular, the sequence F_n with $n = 0, 1, 2, \dots$ is a Følner sequence for G .

3.1 Topological entropy

We endow G with the discrete topology. Let X be a nonempty compact set in a topological space, and let $T : G \times X \rightarrow X$ be a continuous action of G on X . The ordered pair X, T is called a *compact dynamical system*. We now define the topological entropy of X, T .

An *open cover* of X is a set \mathcal{U} of open sets such that $X \subseteq \bigcup \mathcal{U}$. In this case we write

$$C(X, \mathcal{U}) = \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \mathcal{U}, X \subseteq \bigcup \mathcal{F}\}.$$

Note that $C(X, \mathcal{U})$ is a positive integer. If \mathcal{U} and \mathcal{V} are open covers of X , then

$$\text{sup}(\mathcal{U}, \mathcal{V}) = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$$

is again an open cover of X , and

$$C(X, \text{sup}(\mathcal{U}, \mathcal{V})) \leq C(X, \mathcal{U})C(X, \mathcal{V}).$$

For each $g \in G$ and each open cover \mathcal{U} of X , we have another open cover $\mathcal{U}^g = (T^g)^{-1}(\mathcal{U}) = \{(T^g)^{-1}(U) \mid U \in \mathcal{U}\}$. Hence, for each finite set $F \subset G$ we have an open cover $\mathcal{U}^F = \text{sup}\{\mathcal{U}^g \mid g \in F\}$. Let us write $C(X, T, \mathcal{U}, F) = C(X, \mathcal{U}^F)$. Note that $C(X, \mathcal{U}^g) \leq C(X, \mathcal{U})$, hence $C(X, \mathcal{U}^F) \leq C(X, \mathcal{U})^{|F|}$, hence $\log_2 C(X, T, \mathcal{U}, F) \leq |F| \log_2 C(X, \mathcal{U})$. We define

$$\text{ent}(X, T, \mathcal{U}) = \lim_{n \rightarrow \infty} \frac{\log_2 C(X, T, \mathcal{U}, F_n)}{|F_n|} \quad (1)$$

and

$$\text{ent}(X, T) = \sup\{\text{ent}(X, T, \mathcal{U}) \mid \mathcal{U} \text{ is an open cover of } X\}.$$

The nonnegative real number $\text{ent}(X, T)$ is known as the *topological entropy*² of X, T . It measures what might be called the ‘‘asymptotic exponential growth rate’’ of X, T . See for instance [8, 21, 31].

Lemma 3.1. The limit in equation (1) exists.

Proof. Let us write $C_n = C(X, T, \mathcal{U}, F_n)$. Clearly $C_m \leq C_n$ whenever $m \leq n$. Moreover, it is easy to see that $C_{nk} \leq C_n^{k^d}$ for all positive integers k . We are trying to prove that $\log_2 C_n / |F_n|$ approaches a limit as $n \rightarrow \infty$. Assume $G = \mathbb{Z}^d$, so that $|F_n| = (2n + 1)^d$. (The case $G = \mathbb{N}^d$ is similar, with $|F_n| = (n + 1)^d$.)

Fix a positive integer m . Given $n \geq m$, let k be a positive integer such that $mk \leq n < m(k + 1)$. We have $|F_n| \geq |F_{mk}|$ and

$$\frac{|F_{mk}|}{k^d |F_m|} = \left(\frac{2mk + 1}{2mk + k} \right)^d > \left(\frac{2m}{2m + 1} \right)^d$$

²Instead of \log_2 we could use \log_b for any fixed $b > 1$, for instance $b = e$ or $b = 10$. The base $b = 2$ is convenient for information theory, where entropy is measured in bits.

and $\log_2 C_n \leq \log_2 C_{m(k+1)} \leq (k+1)^d \log_2 C_m$, hence

$$\frac{\log_2 C_n}{|F_n|} \leq \frac{(k+1)^d \log_2 C_m}{|F_{mk}|} \leq \frac{(k+1)^d \log_2 C_m}{k^d |F_m|} \left(\frac{2m+1}{2m} \right)^d.$$

As $n \rightarrow \infty$ we have $k \rightarrow \infty$, hence

$$\limsup_{n \rightarrow \infty} \frac{\log_2 C_n}{|F_n|} \leq \frac{\log_2 C_m}{|F_m|} \left(\frac{2m+1}{2m} \right)^d,$$

and this holds for all m , hence

$$\limsup_{n \rightarrow \infty} \frac{\log_2 C_n}{|F_n|} \leq \liminf_{m \rightarrow \infty} \frac{\log_2 C_m}{|F_m|}.$$

In other words, $\lim_{n \rightarrow \infty} \log_2 C_n / |F_n|$ exists, Q.E.D. \square

Let \mathcal{U} and \mathcal{V} be open covers of X . We say that \mathcal{U} *refines* \mathcal{V} if each $U \in \mathcal{U}$ is included in some $V \in \mathcal{V}$. Obviously this implies $C(X, \mathcal{U}) \geq C(X, \mathcal{V})$, and it is also easy to see that $\text{ent}(X, T, \mathcal{U}) \geq \text{ent}(X, T, \mathcal{V})$.

Lemma 3.2. For each m we have $\text{ent}(X, T, \mathcal{U}^{F_m}) = \text{ent}(X, T, \mathcal{U})$.

Proof. Clearly \mathcal{U}^{F_m} refines \mathcal{U} , hence $\text{ent}(X, T, \mathcal{U}^{F_m}) \geq \text{ent}(X, T, \mathcal{U})$. For all n we have $F_{m+n} \supseteq F_m + F_n$, hence $\mathcal{U}^{F_{m+n}}$ refines $\mathcal{U}^{F_m + F_n}$. Moreover, it is easy to check that $\mathcal{U}^{F_m + F_n} \subseteq (\mathcal{U}^{F_m})^{F_n}$, so in particular $\mathcal{U}^{F_m + F_n}$ refines $(\mathcal{U}^{F_m})^{F_n}$. Thus we have $C(X, \mathcal{U}^{F_{m+n}}) \geq C(X, \mathcal{U}^{F_m + F_n}) \geq C(X, (\mathcal{U}^{F_m})^{F_n})$, hence $C(X, T, \mathcal{U}, F_{m+n}) \geq C(X, T, \mathcal{U}^{F_m}, F_n)$, hence

$$\frac{\log_2 C(X, T, \mathcal{U}, F_{m+n})}{|F_n|} \geq \frac{\log_2 C(X, T, \mathcal{U}^{F_m}, F_n)}{|F_n|}$$

hence

$$\frac{\log_2 C(X, T, \mathcal{U}, F_{m+n})}{|F_{m+n}|} \cdot \frac{|F_{m+n}|}{|F_n|} \geq \frac{\log_2 C(X, T, \mathcal{U}^{F_m}, F_n)}{|F_n|}.$$

Taking the limit as $n \rightarrow \infty$ and noting that

$$\lim_{n \rightarrow \infty} \frac{|F_{m+n}|}{|F_n|} = 1,$$

we obtain $\text{ent}(X, T, \mathcal{U}) \geq \text{ent}(X, T, \mathcal{U}^{F_m})$. This completes the proof. \square

Let \mathcal{U} be an open cover of X . We say that \mathcal{U} is a *topological generator* if for each open cover \mathcal{V} of X there exists m such that \mathcal{U}^{F_m} refines \mathcal{V} . The following theorem says that we can use a topological generator to compute $\text{ent}(X, T)$.

Theorem 3.3. If \mathcal{U} is a topological generator, then $\text{ent}(X, T) = \text{ent}(X, T, \mathcal{U})$.

Proof. Let \mathcal{V} be an open cover of X . Since \mathcal{U} is a topological generator, let m be such that \mathcal{U}^{F_m} refines \mathcal{V} . Then $\text{ent}(X, T, \mathcal{U}^{F_m}) \geq \text{ent}(X, T, \mathcal{V})$, so by Lemma 3.2 we have $\text{ent}(X, T, \mathcal{U}) \geq \text{ent}(X, T, \mathcal{V})$. Thus $\text{ent}(X, T, \mathcal{U}) = \text{ent}(X, T)$. \square

3.2 Symbolic dynamics

An important class of dynamical systems are the *symbolic* dynamical systems, also known as *subshifts*. We now present some background material on subshifts. See also [19, §13.10] and [4, 15, 33, 34].

As before, let d be a positive integer, and let G be the additive monoid \mathbb{N}^d or the additive group \mathbb{Z}^d . Let A be a nonempty finite set of symbols. We endow A with the discrete topology. Let $A^G = \{x \mid x : G \rightarrow A\}$. We endow A^G with the product topology. Note that each $x \in A^G$ is a function from G to A . For each finite set $F \subset G$ and each $x \in A^G$ let $x \upharpoonright F$ be the restriction of x to F . Thus $A^F = \{x \upharpoonright F \mid x \in A^G\}$. For each $\sigma \in A^F$ we write $\text{dom}(\sigma) = F$ and $|\sigma| = |F|$ and $[\sigma] = \{x \in A^G \mid x \upharpoonright F = \sigma\}$. Note that $[\sigma]$ is a nonempty clopen set in A^G , and $\{[\sigma] \mid \sigma \in A^F\}$ is a pairwise disjoint covering of A^G . Let $A^* = \bigcup_{n=0}^{\infty} A^{F_n}$ and note that $\{[\sigma] \mid \sigma \in A^*\}$ is a basis for the topology of A^G . For any $T \subseteq A^*$ we write $[T] = \bigcup_{\sigma \in T} [\sigma]$. Thus $[T]$ is an open set in A^G .

The *shift action* of G on A^G is the mapping $S : G \times A^G \rightarrow A^G$ given by $S^g(x)(h) = x(g+h)$ for all $g, h \in G$ and all $x \in A^G$. Thus A^G, S is a compact dynamical system, known as the *full shift*. Since $F_0 = \{0\}^d$ is a singleton set, there is an obvious one-to-one correspondence between A^{F_0} and A , so we identify A^{F_0} with A . The *canonical open covering* of A^G is $\mathcal{U} = \mathcal{U}(A, G) = \{[a] \mid a \in A\}$. For each finite set $F \subset G$ we have $\mathcal{U}^F = \{[\sigma] \mid \sigma \in A^F\}$. By compactness of A^G it follows that \mathcal{U} is a topological generator. Moreover $C(A^G, S, \mathcal{U}, F) = C(A^G, \mathcal{U}^F) = |\mathcal{U}^F| = |A^F| = |A|^{|F|}$, so by Theorem 3.3 we have $\text{ent}(A^G, S) = \text{ent}(A^G, S, \mathcal{U}) = \log_2 |A|$.

A set $X \subseteq A^G$ is said to be *shift-invariant* if $S^g(x) \in X$ for all $g \in G$ and all $x \in X$. A *subshift* is a nonempty, closed, shift-invariant subset of A^G . Each subshift $X \subseteq A^G$ gives rise to a compact dynamical system $X, S \upharpoonright G \times X$. We write $\text{ent}(X) = \text{ent}(X, S \upharpoonright G \times X)$, etc. Since \mathcal{U}^F is a pairwise disjoint covering of A^G , we have $C(X, \mathcal{U}, F) = C(X, \mathcal{U}^F) = |X \upharpoonright F|$ where $X \upharpoonright F = \{x \upharpoonright F \mid x \in X\}$. Since \mathcal{U} is a topological generator, it follows by Theorem 3.3 that

$$\text{ent}(X) = \lim_{n \rightarrow \infty} \frac{\log_2 |X \upharpoonright F_n|}{|F_n|}. \quad (2)$$

Lemma 3.4. We have

$$\lim_{n \rightarrow \infty} |X \upharpoonright F_n| 2^{-s|F_n|} = \begin{cases} 0 & \text{if } s > \text{ent}(X), \\ \infty & \text{if } s < \text{ent}(X). \end{cases}$$

Proof. First suppose $s > \text{ent}(X)$. Fix $\epsilon > 0$ such that $s - \epsilon > \text{ent}(X)$. Equation (2) implies that for all sufficiently large n we have $(s - \epsilon)|F_n| > \log_2 |X \upharpoonright F_n|$, hence $|X \upharpoonright F_n| 2^{-|F_n|s} < 2^{-\epsilon|F_n|}$. Letting $n \rightarrow \infty$ we have $|F_n| \rightarrow \infty$, hence $\lim_{n \rightarrow \infty} |X \upharpoonright F_n| 2^{-s|F_n|} = 0$.

Next, suppose $s < \text{ent}(X)$. Fix $\epsilon > 0$ such that $s + \epsilon < \text{ent}(X)$. Equation (2) implies that for all sufficiently large n we have $(s + \epsilon)|F_n| < \log_2 |X \upharpoonright F_n|$, hence $|X \upharpoonright F_n| 2^{-|F_n|s} > 2^{\epsilon|F_n|}$. Letting $n \rightarrow \infty$ we have $|F_n| \rightarrow \infty$, hence $\lim_{n \rightarrow \infty} |X \upharpoonright F_n| 2^{-s|F_n|} = \infty$. \square

3.3 Hausdorff dimension

Let X be a set in a metric space. The s -dimensional Hausdorff measure of X is defined as

$$\mu_s(X) = \lim_{\epsilon \rightarrow 0} \inf_{\mathcal{E}} \sum_{E \in \mathcal{E}} \text{diam}(E)^s$$

where $\text{diam}(E)$ is the diameter of E . Here \mathcal{E} ranges over coverings of X with the property that $\text{diam}(E) \leq \epsilon$ for all $E \in \mathcal{E}$. The Hausdorff dimension of X is

$$\dim(X) = \inf\{s \mid \mu_s(X) = 0\}.$$

Hausdorff measures and Hausdorff dimension have been widely studied, e.g., in connection with the geometry of fractals [10, 14, 29].

We now define what we mean by the Hausdorff dimension of a subshift. The standard metric on A^G is given by $\rho(x, y) = 2^{-|F_n|}$ where $n = -1, 0, 1, 2, \dots$ is as large as possible such that $x \upharpoonright F_n = y \upharpoonright F_n$. (Recall that $F_{-1} = \emptyset$.) Clearly the standard metric on A^G induces the product topology on A^G . Moreover, the standard metric is an ultrametric, i.e., $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$ for all x, y, z . For any set $X \subseteq A^G$ we define $\dim(X)$ = the Hausdorff dimension of X with respect to the standard metric on A^G .

Lemma 3.5. For all subshifts $X \subseteq A^G$ we have $\text{ent}(X) \geq \dim(X)$.

Proof. For each $E \subseteq A^G$ we have $\text{diam}(E) \leq 2^{-|F_n|}$ if and only if $E \subseteq [\sigma]$ for some $\sigma \in A^{F_n}$. Therefore, in the definition of $\mu_s(X)$ and $\dim(X)$ for an arbitrary set $X \subseteq A^G$, we may safely assume that each E is a basic open set, i.e., $E = [\sigma]$ for some $\sigma \in A^*$. Moreover, for each $\sigma \in A^*$ we have $\text{diam}([\sigma]) = 2^{-|\sigma|}$.

Assume now that X is a subshift, and suppose $s > \text{ent}(X)$. By Lemma 3.4 we have

$$\lim_{n \rightarrow \infty} |X \upharpoonright F_n| 2^{-|F_n|s} = 0. \quad (3)$$

But for each n we have $X \subseteq \bigcup_{x \in X} [x \upharpoonright F_n]$ and $\text{diam}([x \upharpoonright F_n]) = 2^{-|F_n|}$, so (3) implies that $\mu_s(X) = 0$, hence $s \geq \dim(X)$. Since this holds for all $s > \text{ent}(X)$, it follows that $\text{ent}(X) \geq \dim(X)$. \square

Remark 3.6. In §4 we shall prove that for all subshifts $X \subseteq A^G$, $\text{ent}(X) = \dim(X)$. In other words, the topological entropy of a subshift is equal to its Hausdorff dimension with respect to the standard metric. While the special case $G = \mathbb{N}$ is due to Furstenberg [11, Proposition III.1], the general result for $G = \mathbb{N}^d$ or $G = \mathbb{Z}^d$ appears to be new.

3.4 Kolmogorov complexity

We now present some background material on Kolmogorov complexity.

As in §3.2 let $A^* = \bigcup_{n=0}^{\infty} A^{F_n}$. In addition let $\{0, 1\}^*$ be the set of finite sequences of 0's and 1's. For each Turing machine M and each finite sequence $\alpha \in \{0, 1\}^*$, let $M(\alpha)$ be the run of M with input α . A function

$$\Phi : \subseteq \{0, 1\}^* \rightarrow A^*$$

is said to be *partial computable* if there exists a Turing machine M such that for all $\alpha \in \{0, 1\}^*$, $\alpha \in \text{dom}(\Phi)$ if and only if $M(\alpha)$ eventually halts, in which case it halts with output $\Phi(\alpha)$. For each such Φ and each $\xi \in A^*$ let

$$K_\Phi(\xi) = \min(\{|\alpha| \mid \Phi(\alpha) = \xi\} \cup \{\infty\}).$$

A partial computable function $\Psi : \subseteq \{0, 1\}^* \rightarrow A^*$ is said to be *universal* if for each partial computable function $\Phi : \subseteq \{0, 1\}^* \rightarrow A^*$ there exists a constant c such that for all $\xi \in A^*$ we have $K_\Psi(\xi) \leq K_\Phi(\xi) + c$. The existence of such a universal function is easily proved. Fix such a universal function Ψ . For each $\xi \in A^*$ we define the *Kolmogorov complexity* of ξ to be $K(\xi) = K_\Psi(\xi)$. Note that $K(\xi)$ is well defined up to an additive constant, i.e., up to $\pm O(1)$. Here “well defined” means that $K(\xi)$ is independent of the choice of Ψ .

Remark 3.7. Actually the complexity notion K defined above is only one of several variant notions, denoted in [37] as KP, KS, KM, KA, KD. These variants are useful in many contexts [9]. However, for our purposes in this paper, the differences among them are immaterial.

3.5 Effective Hausdorff dimension

We now present some background material concerning the effective or computable variant of Hausdorff dimension. Throughout this paper the words “effective” and “computable” refer to Turing’s theory of computability and unsolvability [30, 36].

We first need some terminology. Recall that a *pseudometric space* is an ordered pair D, ρ where D is a set, ρ is a real-valued function on $D \times D$, and $\rho(a, c) + \rho(b, c) \geq \rho(a, b) = \rho(b, a) \geq 0$ for all $a, b, c \in D$. Within D, ρ we define a *Cauchy sequence* to be a sequence $a_i \in D$, $i = 1, 2, \dots$ such that for all $\epsilon > 0$ there exists n such that $\rho(a_i, a_j) < \epsilon$ for all $i, j > n$. Two Cauchy sequences a_i, b_i , $i = 1, 2, \dots$ are said to be *equivalent* if $\lim_{i \rightarrow \infty} \rho(a_i, b_i) = 0$. The *completion* of D, ρ is the metric space $\widehat{D}, \widehat{\rho}$ where \widehat{D} is the set of equivalence classes and $\widehat{\rho}$ is the metric on \widehat{D} induced by ρ .

A pseudometric space D, ρ is said to be *computable* if D is a computable set of natural numbers and ρ is a computable real-valued function on $D \times D$. An *effective Polish space*³ is the completion of a computable pseudometric space. In this case we define the *basic open sets* in $\widehat{D}, \widehat{\rho}$ to be those of the form

$$B(a, r) = \{x \in \widehat{D} \mid \widehat{\rho}(a, x) < r\}$$

where $a \in D$ and r is a positive rational number. A sequence of basic open sets B_i , $i = 1, 2, \dots$ is said to be *computable* if there exist computable sequences a_i, r_i , $i = 1, 2, \dots$ such that $B_i = B(a_i, r_i)$ for all i . A set $X \subseteq \widehat{D}$ is said to be *effectively closed* if its complement $\widehat{D} \setminus X$ is *effectively open*, i.e., $\widehat{D} \setminus X = \emptyset$

³A.k.a., an *effectively presented complete separable metric space*.

or $\widehat{D} \setminus X = \bigcup_{i=1}^{\infty} B_i$ where $B_i, i = 1, 2, \dots$ is a computable sequence of basic open sets. We say that X is *effectively compact* if it is effectively closed and *effectively totally bounded*, i.e., there exists a computable sequence of finite sets $E_i \subseteq D, i = 1, 2, \dots$, such that $X \subseteq \bigcup_{a \in E_i} B(a, 2^{-i})$ for each i .

Let X be a set in an effective Polish space, and let s be a positive real number. We say that X is *effectively s -null* if there exists a computable double sequence of basic open sets $B_{ni}, n = 1, 2, \dots, i = 1, 2, \dots$, such that for each n we have $X \subseteq \bigcup_{i=1}^{\infty} B_{ni}$ and $\sum_{i=1}^{\infty} \text{diam}(B_{ni})^s \leq 2^{-n}$. The *effective Hausdorff dimension of X* is defined as

$$\text{effdim}(X) = \inf\{s \mid X \text{ is effectively } s\text{-null}\}.$$

It is known that

$$\text{effdim}(X) = \sup_{x \in X} \text{effdim}(\{x\}). \quad (4)$$

In addition, it is known that $\text{effdim}(X) = \dim(X)$ provided X is effectively compact. See for instance Reimann [26] and other works including [20, 27, 28].

Note that the above definitions and remarks apply to the effectively compact Polish space A^G with the standard metric as defined in §3.3. In particular we have $\text{effdim}(X) = \dim(X)$ for all effectively closed sets $X \subseteq A^G$. In §5 below we shall prove that $\text{effdim}(X) = \dim(X)$ for all subshifts $X \subseteq A^G$. This result holds even if X is not effectively closed.

For arbitrary subsets of A^G , the following theorem exhibits a relationship between effective Hausdorff dimension and Kolmogorov complexity. We shall see in Theorem 5.3 that the relationship is closer when X is a subshift.

Theorem 3.8 (Mayordomo's Theorem). For any set $X \subseteq A^G$ we have

$$\text{effdim}(X) = \sup_{x \in X} \liminf_{n \rightarrow \infty} \frac{K(x \upharpoonright F_n)}{|F_n|}.$$

Proof. This follows from (4) together with [9, Theorem 13.3.4]. \square

3.6 Measure-theoretic entropy

We now present some background material on measure-theoretic entropy. We state two important theorems without proof but with references to the literature.

Let X, μ be a probability space. An action $T : G \times X \rightarrow X$ is said to be *measure-preserving* if $\mu((T^g)^{-1}(P)) = \mu(P)$ for each $g \in G$ and each μ -measurable set $P \subseteq X$. In this case the ordered triple X, T, μ is called a *measure-theoretic dynamical system*. We now proceed to define the measure-theoretic entropy of X, T, μ .

A *measurable partition* of X is a finite set \mathcal{P} of pairwise disjoint μ -measurable subsets of X such that $X = \bigcup \mathcal{P}$. In this case we write

$$H(X, \mu, \mathcal{P}) = - \sum_{P \in \mathcal{P}} \mu(P) \log_2 \mu(P).$$

If \mathcal{P} and \mathcal{Q} are measurable partitions of X , then

$$\text{sup}(\mathcal{P}, \mathcal{Q}) = \{P \cap Q \mid P \in \mathcal{P}, Q \in \mathcal{Q}\}$$

is again a measurable partition of X , and it can be shown [8, 10.4(d)] that

$$H(X, \mu, \text{sup}(\mathcal{P}, \mathcal{Q})) \leq H(X, \mu, \mathcal{P}) + H(X, \mu, \mathcal{Q}). \quad (5)$$

For each $g \in G$ and each measurable partition \mathcal{P} of X , we have another measurable partition $\mathcal{P}^g = (T^g)^{-1}(\mathcal{P}) = \{(T^g)^{-1}(P) \mid P \in \mathcal{P}\}$. Hence, for each finite set $F \subset G$ we have a measurable partition $\mathcal{P}^F = \text{sup}\{\mathcal{P}^g \mid g \in F\}$. Let us write $H(X, T, \mu, \mathcal{P}, F) = H(X, \mu, \mathcal{P}^F)$. It follows from (5) that $H(X, T, \mu, \mathcal{P}, F) \leq |F|H(X, Y, \mu, \mathcal{P})$. We define

$$\text{ent}(X, T, \mu, \mathcal{P}) = \lim_{n \rightarrow \infty} \frac{H(X, T, \mu, \mathcal{P}, F_n)}{|F_n|} \quad (6)$$

and

$$\text{ent}(X, T, \mu) = \text{sup}\{\text{ent}(X, T, \mu, \mathcal{P}) \mid \mathcal{P} \text{ is a measurable partition of } X\}.$$

It can be proved that the limit in (6) exists. The nonnegative real number $\text{ent}(X, T, \mu)$ is known as the *measure-theoretic entropy* of X, T, μ . It plays an important role in ergodic theory. See for instance [8, 21, 24].

Let X, T, μ be a measure-theoretic dynamical system. A set $P \subseteq X$ is said to be *G-invariant* if $(T^g)^{-1}(P) \subseteq P$ for all $g \in G$. The system X, T, μ is said to be *ergodic* if for every G -invariant μ -measurable set $P \subseteq X$ we have $\mu(P) = 0$ or $\mu(P) = 1$.

Now let d be a positive integer, let $G = \mathbb{N}^d$ or \mathbb{Z}^d , let A be a nonempty finite set of symbols, and let $X \subseteq A^G$ be a subshift. A Borel probability measure μ on X is said to be *shift-invariant* if $\mu((S^g)^{-1}(P)) = \mu(P)$ for each $g \in G$ and each Borel set $P \subseteq X$. In this case X, S, μ is a measure-theoretic dynamical system, and we write $H(X, \mu, \mathcal{P}) = H(X, S, \mu, \mathcal{P})$, $\text{ent}(X, \mu) = \text{ent}(X, S, \mu)$, etc. As in §3.2 it can be shown that $\text{ent}(X, \mu) = \text{ent}(X, \mu, \mathcal{P})$ where \mathcal{P} is the *canonical measurable partition* of X , namely $\mathcal{P} = \{[a] \cap X \mid a \in A\}$.

In the case of an ergodic subshift, there is the following suggestive characterization of measure-theoretic entropy.

Theorem 3.9 (Shannon/McMillan/Breiman Theorem). Let $X \subseteq A^G$ be a subshift, and let μ be an ergodic, shift-invariant, probability measure on X . Then for μ -almost all $x \in X$ we have

$$\text{ent}(X, \mu) = \lim_{n \rightarrow \infty} \frac{\log_2 \mu([x \upharpoonright F_n])}{-|F_n|}.$$

Proof. See [23]. □

We end this section by noting a significant relationship between topological entropy and measure-theoretic entropy.

Theorem 3.10 (Variational Principle). For any subshift $X \subseteq A^G$ we have

$$\text{ent}(X) = \max_{\mu} \text{ent}(X, \mu)$$

where μ ranges over ergodic, shift-invariant, probability measures on X .

Proof. See [21] and [8, §§16–20]. □

4 Entropy = dimension

As in §3 let d be a positive integer, let $G = \mathbb{N}^d$ or $G = \mathbb{Z}^d$, let A be a finite set of symbols, and let $X \subseteq A^G$ be a subshift. The purpose of this section is to prove that $\text{ent}(X) = \dim(X)$. The special case $G = \mathbb{N}$ is due to Furstenberg [11, Proposition III.1]. However, the general result for $G = \mathbb{N}^d$ or $G = \mathbb{Z}^d$ appears to be new.

As a warm-up for our proof of the general result, we first present Furstenberg's proof of the special case $G = \mathbb{N}$.

Theorem 4.1 (Furstenberg 1967). Let $X \subseteq A^{\mathbb{N}}$ be a one-sided subshift. Then $\text{ent}(X) = \dim(X)$.

Proof. By Lemma 3.5 we have $\text{ent}(X) \geq \dim(X)$. To prove $\text{ent}(X) \leq \dim(X)$ it suffices to prove $\text{ent}(X) \leq s$ for all s such that $\mu_s(X) = 0$. Since $\mu_s(X) = 0$ let \mathcal{E} be such that $X \subseteq \bigcup \mathcal{E}$ and $\sum_{E \in \mathcal{E}} \text{diam}(E)^s < 1$. As noted in §3.3, we may safely assume that each $E \in \mathcal{E}$ is of the form $E = [\sigma]$ where $\sigma \in A^*$, so that $\text{diam}(E) = 2^{-|\sigma|}$. By compactness we may assume that \mathcal{E} is finite. Let us write $\mathcal{E} = \{[\sigma] \mid \sigma \in I\}$ where $I \subset A^*$ is finite. Let $m = \max\{|\sigma| \mid \sigma \in I\}$. From

$$\sum_{\sigma \in I} 2^{-|\sigma|s} = \sum_{E \in \mathcal{E}} \text{diam}(E)^s < 1$$

it follows that

$$\sum_{\sigma_1, \dots, \sigma_k} 2^{-(|\sigma_1| + \dots + |\sigma_k|)s} = \sum_{k=1}^{\infty} \left(\sum_{\sigma \in I} 2^{-|\sigma|s} \right)^k = M < \infty$$

where the sum is taken over all nonempty finite sequences $\sigma_1, \dots, \sigma_k \in I$.

The previous paragraph applies to any subshift. We now bring in the special assumption $G = \mathbb{N}$. Because $G = \mathbb{N}$ and $F_n = \{0, 1, \dots, n\}$, each $x \in A^G$ is an infinite sequence of symbols in A , and each $\sigma \in A^* = \bigcup_{n=0}^{\infty} A^{F_n}$ is a nonempty finite sequence of symbols in A . Thus, given $x \in X$, we can recursively define an infinite sequence $\sigma_1, \dots, \sigma_k, \dots \in I$ such that $S^{|\sigma_1| + \dots + |\sigma_{k-1}|}(x) \in [\sigma_k]$ for all k , and then $x = \sigma_1 \hat{\ } \dots \hat{\ } \sigma_k \hat{\ } \dots$ where $\hat{\ }$ denotes concatenation of finite sequences. Now, given $n \geq 0$, let k be as small as possible such that $x|_{F_n} \subseteq \sigma_1 \hat{\ } \dots \hat{\ } \sigma_k$. We then have

$$|F_n| \leq |\sigma_1| + \dots + |\sigma_k| < |F_n| + m \tag{7}$$

and $[x \upharpoonright F_n] \supseteq [\sigma_1 \wedge \dots \wedge \sigma_k]$. Since the sets $[\xi]$ for $\xi \in X \upharpoonright F_n$ are pairwise disjoint, it follows that $|X \upharpoonright F_n|$ is less than or equal to the number of finite sequences $\sigma_1, \dots, \sigma_k \in I$ such that (7) holds. For each such finite sequence we have $2^{-|F_n|s} < 2^{ms} 2^{-(|\sigma_1| + \dots + |\sigma_k|)s}$, so by summing over all such finite sequences we obtain $|X \upharpoonright F_n| 2^{-|F_n|s} < 2^{ms} M$. Thus $|X \upharpoonright F_n| 2^{-|F_n|s}$ is bounded as $n \rightarrow \infty$. It follows by Lemma 3.4 that $\text{ent}(X) \leq s$, Q.E.D. \square

We now generalize Furstenberg's result.

Theorem 4.2. Let $G = \mathbb{N}^d$ or $G = \mathbb{Z}^d$ where d is a positive integer. Let A be a finite nonempty set of symbols, and let $X \subseteq A^G$ be a subshift. Then $\text{ent}(X) = \dim(X)$.

Proof. By Lemma 3.5 we have $\text{ent}(X) \geq \dim(X)$. To prove $\text{ent}(X) \leq \dim(X)$ it suffices to prove $\text{ent}(X) \leq s$ for all s such that $\mu_s(X) = 0$. Using $\mu_s(X) = 0$ and the compactness of X , we can find finite sets $I_l \subset A^*$ for $l = 1, 2, \dots$ such that $X \subseteq \bigcup_{\sigma \in I_l} [\sigma]$ and $\sum_{\sigma \in I_l} 2^{-|\sigma|s} < 2^{-l}$ and $|\sigma| \ll |\tau|$ for all $\sigma \in I_l$ and all $\tau \in I_{l+1}$. Let $I_\infty = \bigcup_{l=1}^\infty I_l$. We have

$$\sum_{\sigma \in I_\infty} 2^{-|\sigma|s} < \sum_{l=1}^\infty 2^{-l} = 1$$

hence

$$\sum_{\sigma_1, \dots, \sigma_k} 2^{-(|\sigma_1| + \dots + |\sigma_k|)s} = \sum_{k=1}^\infty \left(\sum_{\sigma \in I_\infty} 2^{-|\sigma|s} \right)^k = M < \infty$$

where the first sum is taken over all nonempty finite sequences $\sigma_1, \dots, \sigma_k \in I_\infty$.

Given $\sigma \in I_\infty$ and $g \in G$, let σ^g be the g -shift of σ , i.e., $\text{dom}(\sigma^g) = \{g + h \mid h \in \text{dom}(\sigma)\}$ and $\sigma^g(g + h) = \sigma(h)$ for all $h \in \text{dom}(\sigma)$. Note that $|\sigma^g| = |\sigma|$ and $[\sigma^g] = [\sigma]^g = (S^g)^{-1}([\sigma])$. Since X is a subshift and $X \subseteq \bigcup_{\sigma \in I_l} [\sigma]$ for all l , we have

$$\forall l (\forall g \in G) (\forall x \in X) (\exists \sigma \in I_l) (x \in [\sigma^g]).$$

Let $J_\infty = \bigcup_{l=1}^\infty J_l$ where $J_l = \{\sigma^g \mid \sigma \in I_l, g \in G\}$.

Lemma 4.3. Let $\epsilon > 0$ be given. For all sufficiently large n and each $x \in X$, we can find a pairwise disjoint set $L \subset J_\infty$ such that $\bigcup L \subseteq x \upharpoonright F_n$ and $|\bigcup L| > |F_n|(1 - \epsilon)$ and $|L| < |F_n|\epsilon$.

Proof. This is similar to the Vitali Covering Lemma. Let l be so large that $(3/4)^{ld} < \epsilon$ and $1 < \epsilon|\sigma|$ for all $\sigma \in I_l$. Let n be so large that $n \gg |\sigma|$ for all $\sigma \in I_{2l-1}$. Given $x \in X$ let $\xi = x \upharpoonright F_n$ and let $K_1 = \{\tau \in J_{2l-1} \mid \tau \subset \xi\}$. Since $|\tau| \ll n$ for all $\tau \in K_1$, we have $|\bigcup K_1| \geq (3/4)^d |\xi|$. Let $L_1 \subseteq K_1$ be pairwise disjoint⁴ such that $|\bigcup L_1| \geq |\bigcup K_1|/3^d$. It follows that $|\bigcup L_1| \geq |\xi|/4^d$, hence

⁴Here are the details. Define $L_1 = \{v_j \mid j = 1, 2, \dots\}$ where $v_j \in K_1$ is chosen inductively so that $v_i \cap v_j = \emptyset$ for all $i < j$ and $|v_j|$ is as large as possible. Then for all $\tau \in K_1$ there exists $v \in L_1$ such that $\tau \cap v \neq \emptyset$ and $|\tau| \leq |v|$. From this it follows that $|\bigcup L_1| \geq |\bigcup K_1|/3^d$.

$|\xi \setminus \bigcup L_1| \leq (3/4)^d |\xi|$. If $|\xi \setminus \bigcup L_1| \leq (3/4)^{2d} |\xi|$ let $K_2 = \emptyset$, otherwise let $K_2 = \{\tau \in J_{2l-2} \mid \tau \subset \xi \setminus \bigcup K_1\}$. In the latter case, since $|\tau| \ll |v|$ for all $\tau \in K_2$ and all $v \in L_1$, we have $|\bigcup K_2| \geq (3/4)^d |\xi \setminus \bigcup L_1|$. As before let $L_2 \subseteq K_2$ be pairwise disjoint such that $|\bigcup L_2| \geq |\bigcup K_2|/3^d$. It follows as before that $|\xi \setminus \bigcup(L_1 \cup L_2)| \leq (3/4)^{2d} |\xi|$. Continuing in this fashion for l steps, we obtain pairwise disjoint sets $L_1 \subseteq J_{2l-1}$ and $L_2 \subseteq J_{2l-2}$ and \dots and $L_l \subseteq J_l$ such that $|\xi \setminus \bigcup(L_1 \cup \dots \cup L_l)| \leq (3/4)^{ld} |\xi|$. Finally let $L = L_1 \cup \dots \cup L_l$. Since $J_{2l-1}, J_{2l-2}, \dots, J_l$ are pairwise disjoint, L is pairwise disjoint. We have $|\xi \setminus \bigcup L| \leq (3/4)^{ld} |\xi| < \epsilon |\xi| = |F_n| \epsilon$, hence $|\bigcup L| > (1 - \epsilon) |\xi| = |F_n| (1 - \epsilon)$. By construction we have $\bigcup L \subseteq \xi$, hence $\sum_{\tau \in L} |\tau| = |\bigcup L| \leq |\xi| = |F_n|$. For each $\tau \in L$ we have $1 < \epsilon |\tau|$, hence $|L| < |F_n| \epsilon$. This proves Lemma 4.3. \square

Lemma 4.4. Let ϵ and n be as in Lemma 4.3. Then $|X \upharpoonright F_n|$ is less than or equal to $(|A| + 1)^{2|F_n| \epsilon}$ times the number of sequences $\sigma_1, \dots, \sigma_k \in I_\infty$ such that $|\sigma_k| + \dots + |\sigma_1| \leq |F_n|$.

Proof. Given $x \in X$ let $L = \{\tau_1, \dots, \tau_k\}$ be as in the conclusion of Lemma 4.3. For each $i = 1, \dots, k$ let $\sigma_i \in I_\infty$ be such that $\tau_i = \sigma_i^g$ for some $g \in G$. Since τ_1, \dots, τ_k are pairwise disjoint and $\bigcup_{i=1}^k \tau_i = \bigcup L \subseteq x \upharpoonright F_n$, we have $|\sigma_1| + \dots + |\sigma_k| = |\tau_1| + \dots + |\tau_k| \leq |F_n|$. For each $i = 1, \dots, k$ let g_i be the lexicographically least element of $\text{dom}(\tau_i)$. In other words, g_i is what might be called the ‘‘lower left-hand corner’’ of $\text{dom}(\tau_i)$. Reordering τ_1, \dots, τ_k as necessary, we may assume that $g_1 <_{\text{lex}} \dots <_{\text{lex}} g_k$. Let $U = F_n \setminus \bigcup_{i=1}^k \text{dom}(\tau_i)$ and let $V = U \cup \{g_1, \dots, g_k\}$. By Lemma 4.3 we have $|U| = |F_n| - |\bigcup L| < |F_n| \epsilon$ and $k = |L| < |F_n| \epsilon$, hence $|V| = |U| + k \leq m$ where $m = 2 \lfloor |F_n| \epsilon \rfloor$. For each $j = 1, \dots, m$ define $a_j \in A \cup \{0\}$ as follows. If $j \leq |V|$ let g be the j th element of V with respect to $<_{\text{lex}}$, and let $a_j = x(g)$ if $g \in U$. Otherwise let $a_j = 0$. Clearly $x \upharpoonright F_n$ can be recovered from the pair of sequences a_1, \dots, a_m and $\sigma_1, \dots, \sigma_k$. This proves Lemma 4.4. \square

To prove Theorem 4.2, let ϵ and n be as in Lemmas 4.3 and 4.4. Because $|\sigma_1| + \dots + |\sigma_k| \leq |F_n|$ implies $2^{-|F_n|s} \leq 2^{-(|\sigma_1| + \dots + |\sigma_k|)s}$, it follows from Lemma 4.4 and the definition of M that

$$|X \upharpoonright F_n| 2^{-|F_n|s} < (|A| + 1)^{2|F_n| \epsilon} M,$$

i.e.,

$$|X \upharpoonright F_n| 2^{-|F_n|(s+2\epsilon \log_2(|A|+1))} < M.$$

Thus we see that $|X \upharpoonright F_n| 2^{-|F_n|(s+2\epsilon \log_2(|A|+1))}$ is bounded as n goes to infinity. Hence by Lemma 3.4 we have $\text{ent}(X) \leq s + 2\epsilon \log_2(|A| + 1)$. Since this holds for all $\epsilon > 0$, it follows that $\text{ent}(X) \leq s$. This completes the proof. \square

5 Dimension = complexity

As before let d be a positive integer, let $G = \mathbb{N}^d$ or $G = \mathbb{Z}^d$, let A be a finite set of symbols, and let $X \subseteq A^G$ be a subshift. In this section we prove that

the Hausdorff dimension of X is equal to the effective Hausdorff dimension of X . In addition we obtain a sharp characterization of $\text{dim}(X)$ in terms of the Kolmogorov complexity of finite pieces of the individual orbits of X , i.e., in terms of $K(x \upharpoonright F_n)$ for $x \in X$ and $n = 1, 2, \dots$. Our results apply even when X is not effectively closed.

Lemma 5.1. For all $x \in X$ we have

$$\limsup_{n \rightarrow \infty} \frac{K(x \upharpoonright F_n)}{|F_n|} \leq \text{ent}(X). \quad (8)$$

Proof. Fix $s > \text{ent}(X)$. Let M and I_1, \dots, I_l, \dots and J_1, \dots, J_l, \dots be as in the proof of Theorem 4.2. Fix $\epsilon > 0$. The proofs of Lemmas 4.3 and 4.4 and Theorem 4.2 give the following stronger conclusions.

1. There exists $l \geq 1$ such that for all sufficiently large n and all $x \in X$ there exists a pairwise disjoint set $L \subset J_{2l-1} \cup \dots \cup J_l$ such that $\bigcup L \subseteq x \upharpoonright F_n$ and $|\bigcup L| > |F_n|(1 - \epsilon)$ and $|L| < |F_n|\epsilon$.
2. If l and n are as above, then uniformly in n we can effectively find $C_n \subseteq A^{F_n}$ such that $x \upharpoonright F_n \in C_n$ for all $x \in X$, and $|C_n|$ is less than or equal to $(|A| + 1)^{2|F_n|\epsilon}$ times the number of sequences $\sigma_1, \dots, \sigma_k \in I_{2l-1} \cup \dots \cup I_l$ such that $|\sigma_1| + \dots + |\sigma_k| \leq |F_n|$.
3. If n and C_n are as above, we have $|C_n| \leq 2^{|F_n|s}(|A| + 1)^{2|F_n|\epsilon}M$.

On the other hand, since C_n is recursively enumerable uniformly in n , we can find a constant c such that $K(\xi) \leq \log_2 |C_n| + c$ for all n and all $\xi \in C_n$. Therefore, for all sufficiently large n and all $x \in X$ we have

$$K(x \upharpoonright F_n) \leq |F_n|(s + 2\epsilon \log_2(|A| + 1) + \log_2 M + c).$$

Thus for all $x \in X$ we have

$$\limsup_{n \rightarrow \infty} \frac{K(x \upharpoonright F_n)}{|F_n|} \leq s + 2\epsilon \log_2(|A| + 1).$$

Letting $\epsilon \rightarrow 0$ and $s \rightarrow \text{ent}(X)$ we obtain (8). □

Lemma 5.2. For some $x \in X$ we have

$$\lim_{n \rightarrow \infty} \frac{K(x \upharpoonright F_n)}{|F_n|} = \text{ent}(X). \quad (9)$$

Proof. By the Variational Principle 3.10 let μ be an ergodic, shift-invariant, probability measure on X such that $\text{ent}(X, \mu) = \text{ent}(X)$. Fix $s < \text{ent}(X)$. Let

$$D_n = \{\xi \in A^{F_n} \mid K(\xi) < |F_n|s\}.$$

Clearly $|D_n| \leq 2^{|F_n|s}$. Fix $\epsilon > 0$ such that $s + \epsilon < \text{ent}(X)$, and let

$$T_n = \{\xi \in A^{F_n} \mid \mu([\xi]) < 2^{-|F_n|(s+\epsilon)}\}.$$

The Shannon/McMillan/Breiman Theorem 3.9 tell us that for μ -almost all $x \in X$ and all sufficiently large n we have

$$\frac{\log_2 \mu([x \upharpoonright F_n])}{-|F_n|} > s + \epsilon,$$

i.e., $x \upharpoonright F_n \in T_n$, i.e., $x \in [T_n]$. On the other hand, for each n we have

$$\mu([D_n] \cap [T_n]) = \mu([D_n \cap T_n]) \leq 2^{|F_n|s} 2^{-|F_n|(s+\epsilon)} = 2^{-|F_n|\epsilon}$$

and so

$$\sum_{n=1}^{\infty} \mu([D_n] \cap [T_n]) < \infty.$$

Thus the Borel/Cantelli Lemma tells us that, for μ -almost all x and all sufficiently large n , $x \notin [D_n] \cap [T_n]$. But then it follows that, for μ -almost all x and all sufficiently large n , $x \notin [D_n]$, i.e., $x \upharpoonright F_n \notin D_n$, i.e., $K(x \upharpoonright F_n) \geq |F_n|s$. Since this holds for all $s < \text{ent}(X)$, we now see that (9) holds for μ -almost all $x \in X$. This completes the proof. \square

Theorem 5.3. Let $G = \mathbb{N}^d$ or $G = \mathbb{Z}^d$ where d is a positive integer. Let A be a finite set of symbols, and let $X \subseteq A^G$ be a subshift. Then

$$\text{ent}(X) = \dim(X) = \text{effdim}(X).$$

Moreover

$$\dim(X) \geq \limsup_{n \rightarrow \infty} \frac{K(x \upharpoonright F_n)}{|F_n|}$$

for all $x \in X$, and

$$\dim(X) = \lim_{n \rightarrow \infty} \frac{K(x \upharpoonright F_n)}{|F_n|}$$

for some $x \in X$.

Proof. This follows from Theorems 3.8 and 4.2 and Lemmas 5.1 and 5.2. \square

Questions 5.4.

1. Can we find an “elementary” or “direct” proof of Lemma 5.2? I.e., a proof which does not use measure-theoretic entropy?
2. Is it possible to generalize Theorems 4.2 and 5.3 so as to apply to wider classes of groups or semigroups? For example, do Theorems 4.2 and 5.3 continue to hold if G is an amenable group [38]?
3. Is it possible to generalize Theorems 4.2 and 5.3 so as to apply to scaled entropy and scaled Hausdorff dimension? For example, what about

$$\liminf_{n \rightarrow \infty} \frac{K(x \upharpoonright F_n)}{\sqrt{|F_n|}}?$$

References

- [1] M. Baaz, S.-D. Friedman, and J. Krajíček, editors. *Logic Colloquium '01: Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic, held in Vienna, Austria, August 6–11, 2001*. Number 20 in Lecture Notes in Logic. Association for Symbolic Logic, 2005. VIII + 486 pages.
- [2] S. Bezuglyi and S. Kolyada, editors. *Topics in Dynamics and Ergodic Theory*. Number 310 in London Mathematical Society Lecture Note Series. Cambridge University Press, 2003. VIII + 262 pages.
- [3] Rufus Bowen. Topological entropy for noncompact sets. *Transactions of the American Mathematical Society*, 184:125–136, 1973.
- [4] Mike Boyle. Open problems in symbolic dynamics. In [5], pages 69–118, 2008.
- [5] K. Burns, D. Dolgopyat, and Y. Pesin, editors. *Geometric and Probabilistic Structures in Dynamics*. Number 469 in Contemporary Mathematics. American Mathematical Society, 2008. XVI + 340 pages.
- [6] Vaughn Climenhaga. Bowen’s equation in the non-uniform setting. *Ergodic Theory and Dynamical Systems*, 31(4):1163–1182, 2011.
- [7] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory*. Wiley-Interscience, 2nd edition, 2006. XXIII + 748 pages.
- [8] Manfred Denker, Christian Grillenberger, and Karl Sigmund. *Ergodic Theory on Compact Spaces*. Number 527 in Lecture Notes in Mathematics. Springer-Verlag, 1976. IV + 360 pages.
- [9] Rodney G. Downey and Denis Hirschfeldt. *Algorithmic Randomness and Complexity. Theory and Applications of Computability*. Springer-Verlag, 2010. XXVIII + 855 pages.
- [10] Kenneth Falconer. *Fractal Geometry*. Wiley, 2nd edition, 2003. XXVIII + 337 pages.
- [11] Harry Furstenberg. Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. *Mathematical Systems Theory*, 1:1–49, 1967.
- [12] G. Gallavotti, M. Keane, and D. Ruelle, editors. *International Conference on Dynamical Systems in Mathematical Physics*. Number 40 in Astérisque. Société Mathématique de France, 1976. IV + 192 pages.
- [13] S. S. Goncharov, R. Downey, and H. Ono, editors. *Mathematical Logic in Asia: Proceedings of the 9th Asian Logic Conference, Novosibirsk*. World Scientific Publishing Company, Ltd., 2006. VIII + 328 pages.

- [14] Felix Hausdorff. Dimension und äußeres Maß. *Mathematische Annalen*, 79:157–179, 1919.
- [15] Michael Hochman and Tom Meyerovitch. A characterization of the entropies of multidimensional shifts of finite type. *Annals of Mathematics*, 171:2011–2038, 2010.
- [16] Anatole Katok and Jean-Paul Thouvenot. Slow entropy type invariants and smooth realization of commuting measure-preserving transformations. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 33:323–338, 1997.
- [17] A. N. Kolmogorov. Three approaches to the quantitative definition of information. *Problems of Information and Transmission*, 1(1):1–7, 1965.
- [18] Ming Li and Paul Vitányi. *An Introduction to Kolmogorov Complexity and its Applications*. Graduate Texts in Computer Science. Springer-Verlag, 2nd edition, 1997. XX + 637 pages.
- [19] Douglas Lind and Brian Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, 1995. XVI + 495 pages.
- [20] Joseph S. Miller. Extracting information is hard: a Turing degree of non-integral effective Hausdorff dimension. *Advances in Mathematics*, 226:373–384, 2011.
- [21] Michał Misiurewicz. A short proof of the variational principle for a \mathbb{Z}_+^N -action on a compact space. In [12], pages 147–157, 1976.
- [22] André Nies. *Computability and Randomness*. Oxford University Press, 2009. XV + 433 pages.
- [23] Donald Ornstein and Benjamin Weiss. The Shannon-McMillan-Breiman theorem for a class of amenable groups. *Israel Journal of Mathematics*, 44:53–60, 1983.
- [24] Donald Ornstein and Benjamin Weiss. Entropy and isomorphism theorems for actions of amenable groups. *Journal d’Analyse Mathématique*, 48:1–141, 1987.
- [25] Yakov B. Pesin. *Dimension Theory in Dynamical Systems*. Chicago Lectures in Mathematics. University of Chicago Press, 1997. XI + 304 pages.
- [26] Jan Reimann. *Computability and Fractal Dimension*. PhD thesis, University of Heidelberg, 2004. XI + 130 pages.
- [27] Jan Reimann and Frank Stephan. Effective Hausdorff dimension. In [1], pages 369–385, 2005.
- [28] Jan Reimann and Frank Stephan. On hierarchies of randomness tests. In [13], pages 215–232, 2006.

- [29] C. A. Rogers. *Hausdorff Measures*. Cambridge University Press, 1970. VIII + 179 pages.
- [30] Hartley Rogers, Jr. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, 1967. XIX + 482 pages.
- [31] David Ruelle. Statistical mechanics on a compact set with \mathbb{Z}^ν action satisfying expansiveness and specification. *Transactions of the American Mathematical Society*, 187:237–251, 1973.
- [32] Claude E. Shannon and Warren Weaver. *The Mathematical Theory of Communication*. University of Illinois Press, 1949. V + 117 pages.
- [33] Paul C. Shields. *The Ergodic Theory of Discrete Sample Paths*. Number 13 in Graduate Studies in Mathematics. American Mathematical Society, 1996. XI + 249 pages.
- [34] Stephen G. Simpson. Medvedev degrees of 2-dimensional subshifts of finite type. *Ergodic Theory and Dynamical Systems*, 2012. Preprint, 8 pages, 1 May 2007, accepted for publication.
- [35] Yakov Sinai. Kolmogorov-Sinai entropy. *Scholarpedia*, 4(3):2034, 2009. doi:10.4249/scholarpedia.2034.
- [36] Alan M. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 42:230–265, 1936.
- [37] V. A. Uspensky and A. Shen. Relations between varieties of Kolmogorov complexity. *Mathematical Systems Theory*, 29:271–292, 1996.
- [38] Benjamin Weiss. Actions of amenable groups. In [2], pages 226–262, 2003.