

# Mass Problems and Randomness

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## Abstract

A *mass problem* is a set of Turing oracles. If  $P$  and  $Q$  are mass problems, we say that  $P$  is *weakly reducible* to  $Q$  if every member of  $Q$  Turing computes a member of  $P$ . We say that  $P$  is *strongly reducible* to  $Q$  if every member of  $Q$  Turing computes a member of  $P$  via a fixed Turing functional. The *weak degrees* and *strong degrees* are the equivalence classes of mass problems under weak and strong reducibility, respectively. We focus on the countable distributive lattices  $\mathcal{P}_w$  and  $\mathcal{P}_s$  of weak and strong degrees of mass problems given by nonempty  $\Pi_1^0$  subsets of  $2^\omega$ . Using an abstract Gödel/Rosser incompleteness property, we characterize the  $\Pi_1^0$  subsets of  $2^\omega$  whose associated mass problems are of top degree in  $\mathcal{P}_w$  and  $\mathcal{P}_s$ , respectively. Let  $R$  be the set of Turing oracles which are *random* in the sense of Martin-Löf, and let  $\mathbf{r}$  be the weak degree of  $R$ . We show that  $\mathbf{r}$  is a natural intermediate degree within  $\mathcal{P}_w$ . Namely, we characterize  $\mathbf{r}$  as the unique largest weak degree of a  $\Pi_1^0$  subset of  $2^\omega$  of positive measure. Within  $\mathcal{P}_w$  we show that  $\mathbf{r}$  is meet irreducible, does not join to  $\mathbf{1}$ , and is incomparable with all weak degrees of nonempty thin perfect  $\Pi_1^0$  subsets of  $2^\omega$ . In addition, we present other natural examples of intermediate degrees in  $\mathcal{P}_w$ . We relate these examples to reverse mathematics, computational complexity, and Gentzen-style proof theory.

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## 1 Introduction

Among the principal objects of study in recursion theory going back to the seminal work of Turing [59] and Post [44] have been the upper semilattice  $\mathcal{D}_T$  of all *Turing degrees*, i.e., degrees of unsolvability, and its countable sub-semilattice  $\mathcal{R}_T$  consisting of the *recursively enumerable Turing degrees*, i.e., the Turing degrees of recursively enumerable sets of positive integers. See for instance Sacks [46], Rogers [45], Lerman [36], Soare [56], Odifreddi [42, 43].

A major difficulty or obstacle in the study of  $\mathcal{R}_T$  has been the lack of natural examples. Although it has long been known that  $\mathcal{R}_T$  is infinite and structurally rich, to this day no specific, natural examples of recursively enumerable Turing degrees are known, beyond the two original examples noted by Turing:  $\mathbf{0}' =$  the Turing degree of the Halting Problem, and  $\mathbf{0} =$  the Turing degree of solvable problems. Furthermore,  $\mathbf{0}'$  and  $\mathbf{0}$  are respectively the top and bottom elements of  $\mathcal{R}_T$ . This lack of natural examples, although well known and a major source of frustration, has almost never been discussed in print, but see Rogers [45, Section 9.6]. In any case, the paucity of examples in  $\mathcal{R}_T$  is striking, because it is well known that most other branches of mathematics are motivated and

nurtured by a rich stock of natural examples. Clearly it ought to be of interest to somehow overcome this deficiency in the study of  $\mathcal{R}_T$ .

In recent years it has emerged that there are some natural, important, well-behaved degree structures, closely related to but different from  $\mathcal{R}_T$ , which do not suffer from the above mentioned deficiency. Simpson 1999 [50, 51, 54] called attention to the countable distributive lattices  $\mathcal{P}_w$  and  $\mathcal{P}_s$  of weak and strong degrees of mass problems given by nonempty  $\Pi_1^0$  subsets of  $2^\omega$ , and noted the existence of specific, natural degrees which are intermediate between the top and bottom elements of  $\mathcal{P}_w$  and  $\mathcal{P}_s$ . One of the natural intermediate degrees noted by Simpson was the weak degree  $\mathbf{r}$  of the set of Turing oracles which are random in the sense of Martin-Löf [39]. The study of  $\mathcal{P}_w$  and  $\mathcal{P}_s$  has been continued by Simpson [53, 49], Cenzer/Hinman [9], Simpson/Slaman [55], Binns [3, 4, 5], Binns/Simpson [6], Terwijn [58].

The purpose of the present paper is to elucidate additional properties of previously noted natural degrees in  $\mathcal{P}_w$  and  $\mathcal{P}_s$ , and to present some additional natural degrees in  $\mathcal{P}_w$ . Along the way we give a somewhat leisurely introduction to mass problems in general, and to  $\mathcal{P}_w$  and  $\mathcal{P}_s$  in particular, and we review other known results concerning  $\mathcal{P}_w$  and  $\mathcal{P}_s$ .

In a later paper [48] we shall exhibit a natural embedding of the countable upper semilattice  $\mathcal{R}_T$  into the countable distributive lattice  $\mathcal{P}_w$ . This embedding will be one-to-one and will preserve the top and bottom elements as well as the partial order relation and least upper bound operation from  $\mathcal{R}_T$ . In this way we shall see that  $\mathcal{P}_w$  provides a satisfactory solution to several of the well known difficulties concerning  $\mathcal{R}_T$ .

## 2 Recursion-theoretic preliminaries

In this section we establish notation concerning recursive functionals and Turing degrees.

Throughout this paper we use standard recursion-theoretic notation and concepts from Rogers [45] and Soare [56]. We write  $\omega = \{0, 1, 2, \dots\}$  for the set of natural numbers. We write  $\omega^\omega$  for the space of total functions from  $\omega$  into  $\omega$ . We write  $2^\omega$  for the subspace of  $\omega^\omega$  consisting of the total functions from  $\omega$  into  $\{0, 1\}$ . We sometimes identify a set  $A \subseteq \omega$  with its characteristic function  $\chi_A \in 2^\omega$  given by  $\chi_A(n) = 1$  if  $n \in A$ , 0 if  $n \notin A$ . For  $e, n, m \in \omega$  and  $f \in \omega^\omega$  we write  $\{e\}^f(n) = m$  to mean that the Turing machine with Gödel number  $e$  and oracle  $f$  and input  $n$  eventually halts with output  $m$ . Furthermore,  $\{e\}^f(n) \downarrow$  means that  $\{e\}^f(n)$  is defined, i.e.,  $\exists m (\{e\}^f(n) = m)$ , and  $\{e\}^f(n) \uparrow$  means that  $\{e\}^f(n)$  is undefined, i.e.,  $\neg \exists m (\{e\}^f(n) = m)$ . In the absence of an oracle  $f$ , we write simply  $\{e\}(n) = m$ , etc. For  $P \subseteq \omega^\omega$  we consider *recursive functionals*  $\Phi : P \rightarrow \omega^\omega$  given by  $\Phi(f)(n) = \{e\}^f(n)$  for some  $e \in \omega$  and all  $f \in P$  and  $n \in \omega$ . In particular, a function  $h : \omega \rightarrow \omega$  is said to be *recursive* or *computable* if there exists  $e \in \omega$  such that  $h(n) = \{e\}(n)$  for all  $n \in \omega$ . (The terms “recursive” and “computable” are synonymous.) A set  $A \subseteq \omega$  is said to be *recursively enumerable* if it is the image of a recursive

function, i.e.,  $A = \{m \mid \exists n (h(n) = m)\}$  for some recursive  $h : \omega \rightarrow \omega$ .

For  $f, g \in \omega^\omega$  we write  $f \leq_T g$  to mean that  $f$  is *Turing reducible* to  $g$ , i.e.,  $\exists e \forall n (f(n) = \{e\}^g(n))$ . The *Turing degree* of  $f$ , denoted  $\deg_T(f)$ , is the set of all  $g$  such that  $f \equiv_T g$ , i.e.,  $f \leq_T g$  and  $g \leq_T f$ . The set  $\mathcal{D}_T$  of all Turing degrees is partially ordered by putting  $\deg_T(f) \leq \deg_T(g)$  if and only if  $f \leq_T g$ . Under this partial ordering, the bottom element of  $\mathcal{D}_T$  is  $\mathbf{0} = \{f \in \omega^\omega \mid f \text{ is recursive}\}$ . It is known that  $\mathcal{D}_T$  has no top element. Within  $\mathcal{D}_T$ , the least upper bound of  $\deg_T(f)$  and  $\deg_T(g)$  is given as  $\deg_T(f \oplus g)$  where  $(f \oplus g)(2n) = f(n)$  and  $(f \oplus g)(2n + 1) = g(n)$  for all  $n \in \omega$ . The standard, natural example of a Turing degree  $> \deg_T(f)$  is given by the *Turing jump operator*,  $\deg_T(f) \mapsto \deg_T(f)' = \deg_T(f')$ , where  $f'$  is (the characteristic function of) the *Halting Problem* relative to  $f$ ,  $H^f = \{e \mid \{e\}^f(0) \downarrow\}$ . A Turing degree is said to be *recursively enumerable* if it is  $\deg_T(f)$  where  $f = \chi_A$  is the characteristic function of a recursively enumerable set  $A \subseteq \omega$ . The set of all recursively enumerable Turing degrees is denoted  $\mathcal{R}_T$ . Clearly  $\mathcal{R}_T$  is countable, because there are only countably many recursively enumerable sets. It is known that  $\mathcal{R}_T$  is closed under the least upper bound operation inherited from  $\mathcal{D}_T$ , and that  $\mathbf{0}'$  and  $\mathbf{0}$  are the top and bottom elements of  $\mathcal{R}_T$ . Thus  $\mathcal{R}_T$  is a countable upper semilattice with a top and bottom element.

### 3 Mass problems

A *mass problem* is a subset of  $\omega^\omega$ . The underlying idea here is to view a set  $P \subseteq \omega^\omega$  as a “problem” with a “solution” that does not necessarily exist and is not necessarily unique. The “solutions” of  $P$  are simply the members of  $P$ . In the special case when  $P$  is a singleton set, the “solution” exists and is unique, and the mass problem corresponds to a Turing degree.

In accordance with the conceptual scheme which was explained in the previous paragraph, one makes the following definitions.

**Definition 3.1.** Let  $P$  and  $Q$  be subsets of  $\omega^\omega$ . We say that  $P$  is *weakly reducible* to  $Q$ , written  $P \leq_w Q$ , if for all  $g \in Q$  there exists  $f \in P$  such that  $f \leq_T g$ . Conceptually this means that, given any “solution” of the mass problem  $Q$ , we can use it as an oracle to compute a “solution” of the mass problem  $P$ . The *weak degree* of  $P$ , written  $\deg_w(P)$ , is the set of all  $Q$  such that  $P \equiv_w Q$ , i.e.,  $P \leq_w Q$  and  $Q \leq_w P$ . The set  $\mathcal{D}_w$  of all weak degrees is partially ordered by putting  $\deg_w(P) \leq \deg_w(Q)$  if and only if  $P \leq_w Q$ .

**Remark 3.2.** The concept of weak reducibility goes back to Muchnik [41] and has sometimes been called *Muchnik reducibility*.

**Definition 3.3.** We say that  $P$  is *strongly reducible* to  $Q$ , written  $P \leq_s Q$ , if there exists  $e \in \omega$  such that for all  $g \in Q$  there exists  $f \in P$  such that  $f(n) = \{e\}^g(n)$  for all  $n \in \omega$ . In other words,  $P \leq_s Q$  if and only if there exists a recursive functional  $\Phi : Q \rightarrow P$ . Note that strong reducibility is the uniform variant of weak reducibility. Just as for weak degrees, the *strong degree*

of  $P$ , written  $\text{deg}_s(P)$ , is the set of all  $Q$  such that  $P \equiv_s Q$ , i.e.,  $P \leq_s Q$  and  $Q \leq_s P$ . The set  $\mathcal{D}_s$  of all strong degrees is partially ordered by putting  $\text{deg}_s(P) \leq \text{deg}_s(Q)$  if and only if  $P \leq_s Q$ .

**Remark 3.4.** The concept of strong reducibility goes back to Medvedev [40] and has sometimes been called *Medvedev reducibility*.

**Remark 3.5.** Given  $P, Q \subseteq \omega^\omega$ , a *recursive homeomorphism* of  $P$  onto  $Q$  is a recursive functional  $\Phi : P \rightarrow Q$  mapping  $P$  one-to-one onto  $Q$  such that the inverse functional  $\Phi^{-1} : Q \rightarrow P$  is also recursive. In this case we say that  $P$  and  $Q$  are *recursively homeomorphic*. In addition, let us say that  $P$  is *Turing degree isomorphic* to  $Q$  if  $\{\text{deg}_T(f) \mid f \in P\} = \{\text{deg}_T(g) \mid g \in Q\}$ . Clearly recursive homeomorphism of  $P$  and  $Q$  implies strong equivalence and Turing degree isomorphism, either of which implies weak equivalence. No other implications hold.

**Theorem 3.6.**  $\mathcal{D}_w$  and  $\mathcal{D}_s$  are distributive lattices. They have a bottom element, denoted  $\mathbf{0}$ , and a top element, denoted  $\infty$ .

*Proof.* The least upper bound of  $\text{deg}_w(P)$  and  $\text{deg}_w(Q)$  in  $\mathcal{D}_w$  or in  $\mathcal{D}_s$  is given as  $\text{deg}_w(P \times Q)$  where

$$P \times Q = \{f \oplus g \mid f \in P \text{ and } g \in Q\}.$$

The greatest lower bound of  $\text{deg}_w(P)$  and  $\text{deg}_w(Q)$  in  $\mathcal{D}_w$  is  $\text{deg}_w(P \cup Q)$ , or  $\text{deg}_w(P + Q)$  where

$$P + Q = \{\langle 0 \rangle \hat{\ } f \mid f \in P\} \cup \{\langle 1 \rangle \hat{\ } g \mid g \in Q\}.$$

The greatest lower bound of  $\text{deg}_s(P)$  and  $\text{deg}_s(Q)$  in  $\mathcal{D}_s$  is  $\text{deg}_s(P + Q)$ . It is straightforward to check distributivity. The bottom element of  $\mathcal{D}_w$  and  $\mathcal{D}_s$  is

$$\mathbf{0} = \text{deg}_w(\omega^\omega) = \text{deg}_s(\omega^\omega) = \{P \subseteq \omega^\omega \mid \exists f (f \in P \text{ and } f \text{ is recursive})\}.$$

The top element of  $\mathcal{D}_w$  and  $\mathcal{D}_s$  is  $\infty = \{\emptyset\}$ , where  $\emptyset$  denotes the empty set.  $\square$

**Remark 3.7.** There are obvious, natural embeddings of  $\mathcal{D}_T$  into  $\mathcal{D}_w$  and  $\mathcal{D}_s$  given by  $\text{deg}_T(f) \mapsto \text{deg}_w(\{f\})$  and  $\text{deg}_T(f) \mapsto \text{deg}_s(\{f\})$  respectively. Here  $\{f\}$  is the singleton set whose only member is  $f \in \omega^\omega$ . These embeddings are one-to-one and preserve  $\mathbf{0}$  and the partial order relation and least upper bound operation from  $\mathcal{D}_T$ .

**Remark 3.8.** There is an obvious lattice homomorphism of  $\mathcal{D}_s$  onto  $\mathcal{D}_w$  given by  $\text{deg}_s(P) \mapsto \text{deg}_w(P)$ .

**Remark 3.9.**  $\mathcal{D}_w$  is canonically isomorphic to the lattice of upward closed subsets of  $\mathcal{D}_T$  under the set-theoretic operations of intersection and union. Namely, for each  $P \subseteq \omega^\omega$ , the weak degree  $\text{deg}_w(P) \in \mathcal{D}_w$  gets mapped to the upward closure of  $\{\text{deg}_T(f) \mid f \in P\}$  within  $\mathcal{D}_T$ . It follows that  $\mathcal{D}_w$  is a complete distributive lattice. We do not know of an analogous set-theoretic representation of  $\mathcal{D}_s$ .

**Remark 3.10.** For a survey of general mass problems, see Sorbi [57].

## 4 Recursively bounded $\Pi_1^0$ sets

In this section we present some well known generalities concerning recursively bounded  $\Pi_1^0$  sets and almost recursive functions. The reader is advised to skip most of this section now, and refer to it later as needed.

**Definition 4.1.** A predicate  $R \subseteq \omega^\omega \times \omega$  is said to be *recursive* if

$$\exists e \forall f \forall n (\{e\}^f(n) = 1 \text{ if } R(f, n), \text{ and } \{e\}^f(n) = 0 \text{ if } \neg R(f, n)).$$

A set  $P \subseteq \omega^\omega$  is said to be  $\Pi_1^0$  if there exists a recursive predicate  $R \subseteq \omega^\omega \times \omega$  such that  $P = \{f \mid \forall n R(f, n)\}$ .

**Definition 4.2.** A finite sequence of natural numbers  $\sigma = \langle \sigma(0), \dots, \sigma(k-1) \rangle$  is called a *string* of length  $k$ . We write  $\text{lh}(\sigma) = k$ . The set of all strings is denoted  $\omega^{<\omega}$ . If  $\sigma, \tau$  are strings of length  $k, l$  respectively, then the *concatenation*

$$\sigma \frown \tau = \langle \sigma(0), \dots, \sigma(k-1), \tau(0), \dots, \tau(l-1) \rangle$$

is a string of length  $k+l$ . Note that  $\sigma \subseteq \tau$  if and only if  $\sigma \frown \rho = \tau$  for some  $\rho$ , and this implies  $\text{lh}(\sigma) \leq \text{lh}(\tau)$ . If  $\sigma$  is a string of length  $k$ , then for all  $f \in \omega^\omega$  we have  $\sigma \frown f \in \omega^\omega$  defined by  $(\sigma \frown f)(i) = \sigma(i)$  for  $i < k$ ,  $f(i-k)$  for  $i \geq k$ . Note that  $\sigma \subset f$  if and only if  $\sigma \frown g = f$  for some  $g \in \omega^\omega$ . A *tree* is a set  $T \subseteq \omega^{<\omega}$  such that, for all  $\sigma \subseteq \tau \in T$ ,  $\sigma \in T$ . A *path* through  $T$  is an  $f \in \omega^\omega$  such that  $(\forall \sigma \subset f) (\sigma \in T)$ . The set of all paths through  $T$  is denoted  $[T]$ . We sometimes identify a string  $\sigma$  with its Gödel number  $\#(\sigma) \in \omega$ . A tree  $T$  is said to be *recursive* if  $\{\#(\sigma) \mid \sigma \in T\}$  is recursive, and  $\Pi_1^0$  if  $\{\#(\sigma) \mid \sigma \in T\}$  is  $\Pi_1^0$ .

**Theorem 4.3.**  $P \subseteq \omega^\omega$  is  $\Pi_1^0$  if and only if  $P = [T]$  for some recursive tree  $T$ .

*Proof.* If  $T$  is a recursive tree, we have  $[T] = \{f \mid \forall n R(f, n)\}$  where  $R(f, n)$  is the recursive predicate asserting that  $\langle f(0), \dots, f(n-1) \rangle \in T$ . Thus  $[T]$  is  $\Pi_1^0$ . For the converse, assume that  $P$  is  $\Pi_1^0$  with index  $e$ , i.e., via the Turing machine with Gödel number  $e$ . Thus for all  $f \in \omega^\omega$  we have  $f \in P$  if and only if  $\forall n (\{e\}^f(n) = 1)$ , if and only if  $\forall n (\{e\}^f(n) \neq 0)$ . Let us write  $\{e\}^\sigma(n) = m$  to mean that  $\{e\}^\sigma(n) = m$  via a Turing machine computation using only oracle information from  $\sigma \subset f$  and halting in  $\leq \text{lh}(\sigma)$  steps. Note that the 4-place relation  $\{e\}^\sigma(n) = m$  and the 3-place relation  $\{e\}^\sigma(n) \downarrow$  are primitive recursive. We have  $f \in P$  if and only if  $\forall n (\forall \sigma \subset f) (\{e\}^\sigma(n) \neq 0)$ , if and only if  $(\forall \sigma \subset f) (\forall n \leq \text{lh}(\sigma)) (\{e\}^\sigma(n) \neq 0)$ . Thus  $P = [T]$  where  $T$  is the primitive recursive tree consisting of all strings  $\sigma$  such that  $(\forall n \leq \text{lh}(\sigma)) (\{e\}^\sigma(n) \neq 0)$ .  $\square$

**Theorem 4.4.** If  $P, Q \subseteq \omega^\omega$  are  $\Pi_1^0$  and  $\Phi : P \rightarrow \omega^\omega$  is a recursive functional, then the preimage  $\{f \in P \mid \Phi(f) \in Q\}$  is  $\Pi_1^0$ .

*Proof.* By Theorem 4.3 let  $T$  be a recursive tree such that  $Q = [T]$ . Let  $e$  be an *index* of  $\Phi$ , i.e.,  $\Phi(f)(n) = \{e\}^f(n)$  for all  $f \in P$  and all  $n$ . Given  $f \in P$ , we have  $\Phi(f) \in Q$  if and only if for all  $\sigma \subset f$  and all  $\tau \notin T$  there exists  $n < \text{lh}(\tau)$  such that  $\{e\}^\sigma(n) \neq \tau(n)$ . It follows that  $\{f \in P \mid \Phi(f) \in Q\}$  is  $\Pi_1^0$ .  $\square$

**Definition 4.5.** A set  $P \subseteq \omega^\omega$  is said to be *recursively bounded* if there exists a recursive function  $h \in \omega^\omega$  such that  $f(n) < h(n)$  for all  $f \in P$  and  $n \in \omega$ .

**Remark 4.6.** Any subset of a recursively bounded set is recursively bounded. We shall be concerned with subsets of  $\omega^\omega$  which are recursively bounded and  $\Pi_1^0$ . In particular,  $2^\omega$  is recursively bounded and  $\Pi_1^0$ , and we are especially interested in  $\Pi_1^0$  subsets of  $2^\omega$ . As in Theorem 4.3,  $P$  is a  $\Pi_1^0$  subset of  $2^\omega$  if and only if  $P = [T]$  for some recursive tree  $T \subseteq 2^{<\omega}$ . Here  $2^{<\omega}$  denotes the set of all strings of 0's and 1's. See also Theorem 4.10 and Corollary 4.11 below.

**Theorem 4.7.** *Assume that  $P \subseteq \omega^\omega$  is recursively bounded  $\Pi_1^0$ , and assume that  $\Phi : P \rightarrow \omega^\omega$  is a recursive functional. Then the image  $\{\Phi(f) \mid f \in P\}$  is recursively bounded  $\Pi_1^0$ . Moreover, there exists a total recursive functional  $\Phi^* : \omega^\omega \rightarrow \omega^\omega$  such that  $\Phi^*$  extends  $\Phi$ , i.e.,  $\Phi^*(f) = \Phi(f)$  for all  $f \in P$ .*

*Proof.* The key to the proof is compactness. Let  $h \in \omega^\omega$  be a recursive function such that  $\forall i (\forall f \in P) (f(i) < h(i))$ . Then  $P$  is a closed set in the product space

$$Q_h = \prod_{i \in \omega} \{0, 1, \dots, h(i) - 1\} = \{f \in \omega^\omega \mid \forall i (f(i) < h(i))\}.$$

By general topology,  $Q_h$  is compact. Let  $T$  be a recursive tree such that  $P = [T]$ . Let  $e$  be an index of  $\Phi$ , i.e.,  $\Phi(f)(n) = \{e\}^f(n)$  for all  $f \in P$  and all  $n$ . Then for each  $n$  there is a covering of  $Q_h$  by clopen sets  $\{f \mid \sigma \subset f\}$  where  $\sigma$  is a string such that either  $\{e\}^\sigma(n) \downarrow$  or  $\sigma \notin T$ . By compactness of  $Q_h$ , there exists a finite subcovering. Since  $h$  and  $T$  are recursive, a particular finite subcovering  $\sigma_n^0, \dots, \sigma_n^{k_n}$  can be found effectively. Put

$$h^*(n) = \max \left\{ \{e\}^{\sigma_n^i}(n) + 1 \mid i \leq k_n \text{ and } \sigma_n^i \in T \right\}.$$

Then  $h^* : \omega \rightarrow \omega$  is a recursive function, and  $\Phi(f)(n) < h^*(n)$  for all  $f \in P$  and all  $n$ . Thus  $\{\Phi(f) \mid f \in P\}$  is recursively bounded. For all  $g \in \omega^\omega$  we have  $g \in \{\Phi(f) \mid f \in P\}$  if and only if there is no finite covering of  $Q_h$  by strings  $\sigma$  such that either  $\{e\}^\sigma(n) \downarrow$  and  $\neq g(n)$  for some  $n$ , or else  $\sigma \notin T$ . Thus  $\{\Phi(f) \mid f \in P\}$  is  $\Pi_1^0$ . We have now proved the first part of the lemma. To prove the second part, define a recursive functional  $\Phi^* : \omega^\omega \rightarrow \omega^\omega$  by putting  $\Phi^*(f)(n) = \{e\}^{\sigma_n^i}(n)$  where  $i \leq k_n$  is minimal such that  $\sigma_n^i \subset f$  and  $\sigma_n^i \in T$ , or  $\Phi^*(f)(n) = 0$  if no such  $i$  exists. Clearly  $\Phi^*$  is recursive and extends  $\Phi$ .  $\square$

**Definition 4.8.** In general, suppose that to each  $n \in \omega$  we have effectively associated a finite sequence of ordered pairs  $(\sigma_n^0, m_n^0), \dots, (\sigma_n^{k_n}, m_n^{k_n})$  where  $\sigma_n^i \in \omega^{<\omega}$  and  $m_n^i \in \omega$  for each  $i \leq k_n$ . Define a recursive functional  $\Phi : \omega^\omega \rightarrow \omega^\omega$  by putting  $\Phi(f)(n) = m_n^i$  where  $i \leq k_n$  is minimal such that  $\sigma_n^i \subset f$ , or  $\Phi(f)(n) = 0$  if no such  $i$  exists. Then  $\Phi$  is called a *truth table functional*. For  $f, g \in \omega^\omega$  we say that  $f$  is *truth table reducible* to  $g$ , written  $f \leq_{tt} g$ , if there exists a truth table functional  $\Phi$  such that  $f = \Phi(g)$ . Rogers [45, Chapter 8 and Section 9.6] provides general background on truth table reducibility.

**Corollary 4.9.** *Assume that  $P \subseteq \omega^\omega$  is recursively bounded  $\Pi_1^0$ , and assume that  $\Phi : P \rightarrow \omega^\omega$  is a recursive functional. Then  $\Phi$  can be extended to a truth table functional. In particular,  $\Phi(f) \leq_{tt} f$  for all  $f \in P$ .*

*Proof.* In the proof of Theorem 4.7, note that  $\Phi^*$  is a truth table functional.  $\square$

**Theorem 4.10.** *Let  $P$  be a recursively bounded  $\Pi_1^0$  set. Then  $P$  is recursively homeomorphic to a  $\Pi_1^0$  subset of  $2^\omega$ .*

*Proof.* Given a recursively bounded  $\Pi_1^0$  set  $P \subseteq \omega^\omega$ , put

$$P^* = \{G_f \mid f \in P\} \subseteq 2^\omega$$

where  $G_f =$  (the characteristic function of)  $\{2^m 3^n \mid f(m) = n\}$ . Clearly  $f \mapsto G_f$  is a recursive homeomorphism of  $P$  onto  $P^*$ . By Theorem 4.7,  $P^*$  is  $\Pi_1^0$ .  $\square$

**Corollary 4.11.** *The weak (strong) degrees of nonempty recursively bounded  $\Pi_1^0$  sets are the same as the weak (strong) degrees of nonempty  $\Pi_1^0$  subsets of  $2^\omega$ .*

*Proof.* This is immediate from Theorem 4.10.  $\square$

**Definition 4.12.** Given  $P \subseteq \omega^\omega$ , put

$$\text{Ext}(P) = \{\sigma \mid (\exists f \in P) (\sigma \subset f)\} \subseteq \omega^{<\omega},$$

the set of *extendible nodes* of  $P$ . Note that  $\text{Ext}(P)$  is a tree, and  $[\text{Ext}(P)]$  is the topological closure of  $P$  in  $\omega^\omega$ . In particular, if  $P$  is  $\Pi_1^0$ , then  $P = [\text{Ext}(P)]$ .

**Lemma 4.13.** *Let  $P$  be a recursively bounded  $\Pi_1^0$  set. Then  $\text{Ext}(P)$  is  $\Pi_1^0$ .*

*Proof.* Let  $T$  be a recursive tree such that  $P = [T]$ . Let  $h \in \omega^\omega$  be a recursive function such that  $\forall n (\forall f \in P) (f(n) < h(n))$ . As in the proof of Theorem 4.7, consider the compact space  $Q_h = \{g \mid \forall n (g(n) < h(n))\}$ . For each  $\sigma \in \omega^{<\omega}$ , we have  $\sigma \notin \text{Ext}(P)$  if and only if  $Q_h$  is covered by clopen sets  $\{g \mid \tau \subset g\}$  such that either  $\tau \notin T$  or  $\tau$  is incompatible with  $\sigma$ . In this case, compactness of  $Q_h$  implies the existence of a finite subcovering. Moreover, since  $h$  and  $T$  are recursive, such a finite subcovering can be found effectively. Thus  $\{\#\sigma \mid \sigma \notin \text{Ext}(P)\}$  is recursively enumerable, i.e.,  $\Sigma_1^0$ . It follows that  $\text{Ext}(P)$  is  $\Pi_1^0$ .  $\square$

**Definition 4.14.** For  $P \subseteq \omega^\omega$ , an *isolated point* of  $P$  is an  $f \in P$  such that, for some string  $\tau$ ,  $f$  is the unique  $g \in P$  such that  $\tau \subset g$ . We say that  $P$  is *perfect* if  $P$  has no isolated points.

**Theorem 4.15.** *Let  $P$  be a recursively bounded  $\Pi_1^0$  set. If  $f$  is an isolated point of  $P$ , then  $f$  is recursive.*

*Proof.* By Theorem 4.10 we may assume that  $P$  is a  $\Pi_1^0$  subset of  $2^\omega$ . Let  $\tau \in 2^{<\omega}$  be such that  $f$  is the unique  $g \in P$  such that  $\tau \subset g$ . Then, for all  $\sigma \supseteq \tau$  in  $2^{<\omega}$ , we have  $\sigma \subset f$  if and only if  $\sigma \in \text{Ext}(P)$ . By Lemma 4.13,  $\text{Ext}(P)$  is  $\Pi_1^0$ , hence  $A = 2^{<\omega} \setminus \text{Ext}(P)$  is recursively enumerable. Now, given  $\sigma \in 2^{<\omega}$  of length  $n$ , we have  $\sigma \subset f$  if and only if  $\rho \in A$  for all  $\rho \in 2^{<\omega}$  of length  $n$  other than  $\sigma$ . Since  $\{\rho \in 2^{<\omega} \mid \text{lh}(\rho) = n\}$  is of cardinality  $2^n$ , it follows that  $\{\sigma \mid \sigma \subset f\}$  is recursively enumerable, so  $f$  is recursive.  $\square$

**Corollary 4.16.** *Let  $P$  be a recursively bounded  $\Pi_1^0$  set. If the weak or strong degree of  $P$  is  $> \mathbf{0}$ , then  $P$  is perfect.*

*Proof.* For any  $P \subseteq \omega^\omega$ , we have  $\deg_w(P) > \mathbf{0}$  or  $\deg_s(P) > \mathbf{0}$  if and only if  $P$  has no recursive members. If  $P$  is recursively bounded  $\Pi_1^0$  and has no recursive members, then by Theorem 4.15  $P$  has no isolated points, i.e.,  $P$  is perfect.  $\square$

**Definition 4.17.** We say that  $g \in \omega^\omega$  is *almost recursive* if for all  $f \leq_T g$  there exists  $h \in \omega^\omega$  such that  $\forall n (f(n) < h(n))$  and  $h$  is recursive. (Turing degrees which contain almost recursive functions have been known in the literature as *hyperimmune-free* Turing degrees.)

**Theorem 4.18.** *Suppose  $g$  is almost recursive. Then for all  $f \leq_T g$  we have  $f \leq_{tt} g$ , i.e.,  $f$  is truth table reducible to  $g$ . In particular,  $f = \Phi(g)$  for some total recursive functional  $\Phi : \omega^\omega \rightarrow \omega^\omega$ .*

*Proof.* Let  $e$  be such that  $f(n) = \{e\}^g(n)$  for all  $n$ . Define  $f^* \in \omega^\omega$  by  $f^*(n) =$  the least  $k$  such that  $\{e\}^\tau(n) \downarrow$  where  $\tau = \langle g(0), \dots, g(k) \rangle$ . Clearly  $f^* \leq_T g$ . Since  $g$  is almost recursive, there exists a recursive function  $h$  such that  $\forall n (f^*(n) < h(n))$ . Define  $\Phi : \omega^\omega \rightarrow \omega^\omega$  by putting  $\Phi(\bar{g})(n) = \{e\}^{\bar{\tau}}(n)$  where  $\bar{\tau} = \langle \bar{g}(0), \dots, \bar{g}(h(n)) \rangle$ , if  $\{e\}^{\bar{\tau}}(n) \downarrow$ , and  $\Phi(\bar{g})(n) = 0$  otherwise, for all  $\bar{g} \in \omega^\omega$  and  $n \in \omega$ . Then  $\Phi$  is a truth table functional, and  $f = \Phi(g)$ .  $\square$

We end this section with the Almost Recursive Basis Theorem.

**Theorem 4.19.** *If  $P \subseteq \omega^\omega$  is nonempty, recursively bounded, and  $\Pi_1^0$ , then there exists  $g \in P$  such that  $g$  is almost recursive.*

*Proof.* This is the Hyperimmune-Free Basis Theorem of Jockusch/Soare [26, Theorem 2.4]. For completeness we present the proof here. Define inductively a sequence of nonempty  $\Pi_1^0$  sets  $P = P_0 \supseteq P_1 \supseteq \dots \supseteq P_e \supseteq P_{e+1} \supseteq \dots$  as follows. Put  $P_0 = P$ . If  $\exists n (\exists f \in P_e) (\{e\}^f(n) \uparrow)$ , fix such an  $n$  and put  $P_{e+1} = \{f \in P_e \mid \{e\}^f(n) \uparrow\}$ . Otherwise, Theorem 4.7 gives us a recursive function  $h = h_e$  such that  $\forall n (\forall f \in P_e) (\{e\}^f(n) < h(n))$ , and in this case we put  $P_{e+1} = P_e$ . By compactness,  $\bigcap_{e=0}^\infty P_e$  is nonempty, so let  $g \in \bigcap_{e=0}^\infty P_e$ . By construction,  $g$  is almost recursive.  $\square$

## 5 The lattices $\mathcal{P}_w$ and $\mathcal{P}_s$

In this section we introduce the lattices  $\mathcal{P}_w$  and  $\mathcal{P}_s$  which are the focus of this paper.

**Remark 5.1.** There is a large recursion-theoretic literature concerning Turing degrees of members of  $\Pi_1^0$  subsets of  $\omega^\omega$ , and especially Turing degrees of members of recursively bounded  $\Pi_1^0$  subsets of  $\omega^\omega$ . See for instance the classic paper of Jockusch and Soare [26] and the survey article by Cenzer and Remmel [10]. Mindful of this literature, we find it natural to view nonempty recursively bounded  $\Pi_1^0$  sets as mass problems.

By Theorem 4.10 and Corollary 4.11, it suffices to consider  $\Pi_1^0$  subsets of  $2^\omega$ .

**Definition 5.2.**  $\mathcal{P}_w$  ( $\mathcal{P}_s$ ) is the set of weak (strong) degrees of nonempty recursively bounded  $\Pi_1^0$  sets. By Corollary 4.11,  $\mathcal{P}_w$  ( $\mathcal{P}_s$ ) is the same as the set of weak (strong) degrees of nonempty  $\Pi_1^0$  subsets of  $2^\omega$ .

**Theorem 5.3.**  $\mathcal{P}_w$  and  $\mathcal{P}_s$  are countable distributive lattices with a top and bottom element, denoted  $\mathbf{1}$  and  $\mathbf{0}$  respectively.

*Proof.* If  $P, Q \subseteq \omega^\omega$  are  $\Pi_1^0$  and recursively bounded, then so are  $P \cap Q$ ,  $P \cup Q$ ,  $P \times Q$ , and  $P + Q$ . In particular,  $\mathcal{P}_w$  and  $\mathcal{P}_s$  are closed under the least upper bound and greatest lower bound operations inherited from the distributive lattices  $\mathcal{D}_w$  and  $\mathcal{D}_s$ , respectively. It follows that  $\mathcal{P}_w$  and  $\mathcal{P}_s$  are distributive lattices. Clearly  $\mathcal{P}_w$  and  $\mathcal{P}_s$  are countable, because there are only countably many  $\Pi_1^0$  subsets of  $2^\omega$ . Clearly  $\mathbf{0} = \deg_w(2^\omega) = \deg_s(2^\omega)$  is the bottom element of  $\mathcal{P}_w$  and of  $\mathcal{P}_s$ . It remains to show that  $\mathcal{P}_w$  and  $\mathcal{P}_s$  have a top element. Let PA be the set of completions of Peano Arithmetic. Identifying sentences with their Gödel numbers, we may view PA as a  $\Pi_1^0$  subset of  $2^\omega$ . Since Peano Arithmetic is consistent, PA is nonempty. It is known that every nonempty  $\Pi_1^0$  subset of  $2^\omega$  is  $\leq_s$  PA, hence  $\leq_w$  PA. We shall obtain this result and much more in Section 6 below, but see Jockusch/Soare [26]. Let us use  $\mathbf{1}$  ambiguously to denote either  $\deg_w(\text{PA})$  or  $\deg_s(\text{PA})$ . Thus  $\mathbf{1}$  is the top element both of  $\mathcal{P}_w$  and of  $\mathcal{P}_s$ .  $\square$

**Remark 5.4.** There is an obvious lattice homomorphism of  $\mathcal{P}_s$  onto  $\mathcal{P}_w$  given by  $\deg_s(P) \mapsto \deg_w(P)$ . Simpson and Slaman [55] have shown that every nonzero weak degree in  $\mathcal{P}_w$  contains infinitely many strong degrees in  $\mathcal{P}_s$ .

**Remark 5.5.** In the context of recursively bounded  $\Pi_1^0$  sets, there is reason to view weak reducibility as the mass problem analog of Turing reducibility, while strong reducibility is the mass problem analog of truth table reducibility. See Rogers [45, Sections 8.3 and 9.6] and Simpson [54, Remark 3.12]. Namely, if  $Q$  is recursively bounded  $\Pi_1^0$  and  $P \leq_s Q$ , then by Corollary 4.9 the recursive functional  $\Phi : Q \rightarrow P$  is given by truth tables, hence for each  $g \in Q$  there exists  $f = \Phi(g) \in P$  such that  $f \leq_{tt} g$ , i.e.,  $f$  is truth table reducible to  $g$ . Thus we see that  $\mathcal{P}_w$  is analogous to  $\mathcal{R}_T$ , the recursively enumerable Turing degrees, while  $\mathcal{P}_s$  is more closely analogous to  $\mathcal{R}_{tt}$ , the recursively enumerable truth table degrees.

**Remark 5.6.** It is known that the countable distributive lattices  $\mathcal{P}_w$  and  $\mathcal{P}_s$  are structurally rich. Binns/Simpson [3, 6] have shown that every countable distributive lattice is lattice embeddable in every nontrivial initial segment of  $\mathcal{P}_w$ . A similar conjecture for  $\mathcal{P}_s$  remains open, although partial results in this direction are known. Binns [3, 4] has obtained the  $\mathcal{P}_w$  and  $\mathcal{P}_s$  analogs of the Sacks Splitting Theorem. Namely, for all  $\mathbf{b} > \mathbf{0}$  in  $\mathcal{P}_w$  there exist  $\mathbf{b}_1, \mathbf{b}_2 < \mathbf{b}$  in  $\mathcal{P}_w$  such that  $\sup(\mathbf{b}_1, \mathbf{b}_2) = \mathbf{b}$ , and similarly for  $\mathcal{P}_s$ . Cenzer/Hinman [9] have obtained the  $\mathcal{P}_s$  analog of the Sacks Density Theorem. Namely, for all  $\mathbf{a} < \mathbf{b}$  in  $\mathcal{P}_s$  there exists  $\mathbf{c}$  in  $\mathcal{P}_s$  such that  $\mathbf{a} < \mathbf{c} < \mathbf{b}$ . A similar conjecture for  $\mathcal{P}_w$  remains open. Binns [3, 4] has improved the result of Cenzer/Hinman [9] by showing that for all  $\mathbf{a} < \mathbf{b}$  in  $\mathcal{P}_s$  there exist  $\mathbf{b}_1, \mathbf{b}_2 < \mathbf{b}$  in  $\mathcal{P}_s$  such that  $\mathbf{a} < \inf(\mathbf{b}_1, \mathbf{b}_2) < \sup(\mathbf{b}_1, \mathbf{b}_2) = \mathbf{b}$ . These structural results for  $\mathcal{P}_w$  and  $\mathcal{P}_s$

are proved by means of priority arguments. They invite comparison with the older, known results for recursively enumerable Turing degrees, which were also proved by priority arguments.

## 6 Weak and strong completeness

In this section we obtain additional information concerning sets which are of weak or strong degree  $\mathbf{1}$  in  $\mathcal{P}_w$  or  $\mathcal{P}_s$ , respectively. We show that, if  $P$  is a nonempty recursively bounded  $\Pi_1^0$  set, then  $\deg_w(P) = \mathbf{1}$  if and only if  $P$  is Turing degree isomorphic to PA, and  $\deg_s(P) = \mathbf{1}$  if and only if  $P$  is recursively homeomorphic to PA.

By Theorem 4.10 and Corollary 4.11, it suffices to consider  $\Pi_1^0$  subsets of  $2^\omega$ .

**Definition 6.1.** Let  $P$  be a nonempty recursively bounded  $\Pi_1^0$  set.  $P$  is *weakly complete* if  $\deg_w(P) = \mathbf{1}$ , i.e.,  $P \geq_w Q$  for all nonempty  $\Pi_1^0$  sets  $Q \subseteq 2^\omega$ .  $P$  is *strongly complete* if  $\deg_s(P) = \mathbf{1}$ , i.e.,  $P \geq_s Q$  for all nonempty  $\Pi_1^0$  sets  $Q \subseteq 2^\omega$ . These notions have sometimes been referred to as *Muchnik completeness* and *Medvedev completeness*, respectively.

**Remark 6.2.** We have seen in the proof of Theorem 5.3 that PA is strongly complete, hence weakly complete. In addition, there are natural examples of recursively bounded  $\Pi_1^0$  sets which are weakly complete but not strongly complete. See Definition 7.9 and Remark 7.10 below.

**Theorem 6.3.** Let  $P$  and  $Q$  be nonempty  $\Pi_1^0$  subsets of  $2^\omega$ .

1. If  $P$  and  $Q$  are strongly complete, then  $P$  is recursively homeomorphic to  $Q$ .
2. If  $P$  is strongly complete, then we can find a recursive functional  $\Phi : P \rightarrow Q$  which is onto  $Q$ , i.e.,  $Q = \{\Phi(f) \mid f \in P\}$ .
3.  $P$  is strongly complete if and only if  $P$  is productive, i.e., given an index of a nonempty  $\Pi_1^0$  set  $P' \subseteq P$  we can effectively find a canonically indexed clopen set  $U \subseteq 2^\omega$  such that both  $P' \cap U$  and  $P' \setminus U$  are nonempty.

*Proof.* See Simpson [54, Section 3]. □

**Corollary 6.4.** Let  $P$  be a nonempty recursively bounded  $\Pi_1^0$  set. If  $P$  is strongly complete, then the set of Turing degrees of members of  $P$  is upward closed.

*Proof.* For any  $P$ , the set of Turing degrees of members of  $P \times 2^\omega$  is obviously upward closed. Now assume that  $P$  is strongly complete. Then clearly  $P \times 2^\omega$  is strongly complete. Hence, by Theorem 4.10 and part 1 of Theorem 6.3,  $P$  and  $P \times 2^\omega$  are recursively homeomorphic to each other. Since the set of Turing degrees of members of  $P \times 2^\omega$  is upward closed, it follows that the set of Turing degrees of members of  $P$  is upward closed. □

**Corollary 6.5.** *Let  $P$  be a nonempty recursively bounded  $\Pi_1^0$  set. Then  $P$  is strongly complete if and only if  $P$  is recursively homeomorphic to PA.*

*Proof.* Recall that PA is the set of completions of Peano Arithmetic. By the Gödel/Rosser Theorem for Peano Arithmetic, PA is productive. Hence, by part 3 of Theorem 6.3, PA is strongly complete. Our corollary now follows by Theorem 4.10 and part 1 of Theorem 6.3.  $\square$

The next corollary is originally due to Robert M. Solovay.

**Corollary 6.6.** *The set of Turing degrees of members of PA is upward closed.*

*Proof.* This is immediate from Corollaries 6.4 and 6.5.  $\square$

**Remark 6.7.** Instead of Peano Arithmetic, we could have used any consistent recursively axiomatizable theory  $T$  which is *effectively essentially incomplete*, i.e., has the property given by the Gödel/Rosser Theorem. The required property of  $T$  is as follows. Given a consistent recursively axiomatizable theory  $T'$  extending  $T$ , we can effectively find a sentence  $\varphi$  in the language of  $T$  which is *independent of  $T'$* , i.e.,  $T' \not\vdash \varphi$  and  $T' \not\vdash \neg\varphi$ . Compare this with our notion of productivity from part 3 of Theorem 6.3, which may be viewed as an abstract Gödel/Rosser property for  $\Pi_1^0$  subsets of  $2^\omega$ .

**Theorem 6.8.** *Let  $P$  be a nonempty recursively bounded  $\Pi_1^0$  set. Then  $P$  is weakly complete if and only if  $P$  is Turing degree isomorphic to PA.*

*Proof.* By Corollary 6.5, PA is strongly complete. (This is the only property of PA which we shall need.) Hence PA is weakly complete, so any  $P$  which is Turing degree isomorphic to PA is weakly complete. For the converse, let  $P$  be a nonempty  $\Pi_1^0$  subset of  $2^\omega$  which is weakly complete. In particular  $\text{PA} \leq_w P$ . It follows by Corollary 6.6 that the Turing degrees of members of  $P$  are included in the Turing degrees of members of PA.

In order to finish the proof of Theorem 6.8, we need the following lemma, which exposes an interesting relationship between weak reducibility and strong reducibility.

**Lemma 6.9.** *Let  $P$  and  $Q$  be nonempty recursively bounded  $\Pi_1^0$  sets. If  $P \leq_w Q$ , then we can find a nonempty  $\Pi_1^0$  set  $\overline{Q} \subseteq Q$  such that  $P \leq_s \overline{Q}$ .*

*Proof.* By the Almost Recursive Basis Theorem 4.19, let  $g \in Q$  be almost recursive. Since  $P \leq_w Q$ , let  $f \in P$  be such that  $f \leq_T g$ . By Theorem 4.18 we can find a total recursive functional  $\Phi : \omega^\omega \rightarrow \omega^\omega$  such that  $f = \Phi(g)$ . Put  $\overline{Q} = \{\overline{g} \in Q \mid \Phi(\overline{g}) \in P\}$ . By Theorem 4.4 we have that  $\overline{Q}$  is a  $\Pi_1^0$  subset of  $Q$ . Since  $f = \Phi(g) \in P$ , we have  $\overline{g} \in \overline{Q}$ , hence  $\overline{Q}$  is nonempty. Putting  $\overline{\Phi} =$  the restriction of  $\Phi$  to  $\overline{Q}$ , we have  $\overline{\Phi} : \overline{Q} \rightarrow P$ , so  $P \leq_s \overline{Q}$ .  $\square$

Now, since our  $P$  is  $\geq_w$  PA, apply Lemma 6.9 to get a nonempty  $\Pi_1^0$  set  $\overline{P} \subseteq P$  such that  $\overline{P} \geq_s \text{PA}$ . Since PA is strongly complete,  $\overline{P}$  is strongly complete. Hence, by Theorem 4.10 and part 1 of Theorem 6.3,  $\overline{P}$  is recursively

homeomorphic to PA. It follows that  $\overline{P}$  is Turing degree isomorphic to PA. We now have  $\{\deg_T(f) \mid f \in P\} \subseteq \{\deg_T(f) \mid f \in \text{PA}\} = \{\deg_T(f) \mid f \in \overline{P}\} \subseteq \{\deg_T(f) \mid f \in P\}$ , so  $P$  is Turing degree isomorphic to PA. This completes the proof of Theorem 6.8.  $\square$

**Corollary 6.10.** *Let  $P$  and  $Q$  be nonempty recursively bounded  $\Pi_1^0$  sets. If  $P$  and  $Q$  are weakly complete, then  $P$  is Turing degree isomorphic to  $Q$ .*

*Proof.* By Theorem 6.8,  $P$  and  $Q$  are Turing degree isomorphic to PA, hence to each other.  $\square$

We can now strengthen Corollary 6.4 as follows.

**Corollary 6.11.** *Let  $P$  be a nonempty recursively bounded  $\Pi_1^0$  set. If  $P$  is weakly complete, then the set of Turing degrees of members of  $P$  is upward closed.*

*Proof.* This is immediate from Corollary 6.6 and Theorem 6.8.  $\square$

## 7 $\Pi_1^0$ sets of positive measure

**Definition 7.1.** The *fair coin probability measure* on  $2^\omega$  is defined by

$$\mu(\{f \in 2^\omega \mid f(n) = m\}) = \frac{1}{2}$$

for all  $m \in \{0, 1\}$  and  $n \in \omega$ . A set  $P \subseteq 2^\omega$  is said to be of *positive measure* if  $\mu(P) > 0$ .

In this section we prove a “non-helping” theorem for weak and strong degrees of subsets of  $2^\omega$  which are of positive measure.

**Lemma 7.2.** *Let  $F_n$ ,  $n \in \omega$  be a sequence of finite subsets of  $\omega$  of bounded cardinality. Put*

$$S = \prod_{n \in \omega} F_n = \{f \in \omega^\omega \mid \forall n (f(n) \in F_n)\} \subseteq \omega^\omega.$$

*Let  $P \subseteq 2^\omega$  be of positive measure. Let  $Q \subseteq \omega^\omega$  be arbitrary. If  $S \leq_s P \times Q$ , then  $S \leq_s Q$ .*

*Proof.* We generalize an argument of Jockusch/Soare [26, Theorem 5.3]. Let  $k \geq 2$  be such that, for all  $n$ ,  $F_n$  is of cardinality  $< k$ . Our hypothesis concerning  $P$  is that  $\mu(P) > 0$ . By measure theory, let  $V \supseteq P$  be an open set in  $2^\omega$  such that  $\mu(V \setminus P) < \mu(P)/4k$ . Let  $U \subseteq V$  be a clopen set such that  $\mu(V \setminus U) < \mu(P)/4k$ . It follows that  $\mu(U \setminus P) < \mu(U)/k$ . Note that  $\mu(U)$  is a positive rational number. Since  $S \leq_s P \times Q$ , let  $\Phi$  be a recursive functional such that  $\Phi(f \oplus g) \in S$  for all  $f \in P$  and  $g \in Q$ . Given  $g \in Q$  and  $n \in \omega$ , we can effectively find  $m = \Psi(g)(n) \in \omega$  such that  $\mu(\{f \in U \mid \Phi(f \oplus g)(n) = m\}) > \mu(U)/k$ . It follows that  $m \in F_n$ . Thus  $\Psi$  is a recursive functional, and  $\Psi(g) \in S$  for all  $g \in Q$ . Hence  $S \leq_s Q$ .  $\square$

**Lemma 7.3.** *Same as Lemma 7.2 with strong reducibility,  $\leq_s$ , replaced by weak reducibility,  $\leq_w$ .*

*Proof.* Assume  $S \leq_w P \times Q$ . Fix  $g \in Q$ . We have  $S \leq_w P \times \{g\}$ . By countable additivity of  $\mu$ , since there are only countably many recursive functionals, there exists  $P_g \subseteq P$  such that  $\mu(P_g) > 0$  and  $S \leq_s P_g \times \{g\}$ . By Lemma 7.2 it follows that  $S \leq_s \{g\}$ . This implies  $S \leq_w Q$ , since  $g \in Q$  is arbitrary.  $\square$

**Definition 7.4.** For  $A, B \subseteq \omega$  we say that  $f \in 2^\omega$  *separates*  $A, B$  if  $f(n) = 1$  for all  $n \in A$ , and  $f(n) = 0$  for all  $n \in B$ . A nonempty  $\Pi_1^0$  set  $S \subseteq 2^\omega$  is said to be *separating* if there exist recursively enumerable sets  $A, B \subseteq \omega$  such that  $S = \{f \in 2^\omega \mid f \text{ separates } A, B\}$ . In this case we say that the weak degree  $\text{deg}_w(S)$  and the strong degree  $\text{deg}_s(S)$  are *separating*.

**Theorem 7.5.** *Let  $S, P, Q$  be  $\Pi_1^0$  subsets of  $2^\omega$ . Assume that  $S$  is separating and that  $P$  is of positive measure. Let  $\mathbf{s}, \mathbf{p}, \mathbf{q} \in \mathcal{P}_w$  be the weak degrees of  $S, P, Q$  respectively. If  $\mathbf{s} \leq \sup(\mathbf{p}, \mathbf{q})$ , then  $\mathbf{s} \leq \mathbf{q}$ . The same holds for strong degrees.*

*Proof.* It suffices to note that  $S$  is of the form required by Lemmas 7.2 and 7.3. Namely,  $S = \prod_{n \in \omega} F_n$  where  $F_n = \{1\}$  if  $n \in A$ ,  $\{0\}$  if  $n \in B$ ,  $\{0, 1\}$  otherwise.  $\square$

**Corollary 7.6.** *Let  $P$  and  $Q$  be nonempty  $\Pi_1^0$  subsets of  $2^\omega$ . Assume that  $P$  is of positive measure. Let  $\mathbf{p}, \mathbf{q} \in \mathcal{P}_w$  be the weak degrees of  $P, Q$  respectively. If  $\mathbf{q} < \mathbf{1}$ , then  $\sup(\mathbf{p}, \mathbf{q}) < \mathbf{1}$ . The same holds for strong degrees.*

*Proof.* By Theorem 7.5, it suffices to note that  $\mathbf{1}$  is separating. Namely,  $\mathbf{1}$  is the weak or strong degree of the  $\Pi_1^0$  set  $S = \{f \in 2^\omega \mid f \text{ separates } A, B\}$  where  $A = \{n \mid \{n\}(n) = 0\}$  and  $B = \{n \mid \{n\}(n) = 1\}$ . Or, we could take  $A$  and  $B$  to be the set of Gödel numbers of provable and refutable sentences of Peano Arithmetic. See also Jockusch/Soare [26] and Simpson [54, Section 3].  $\square$

**Corollary 7.7.** *Let  $P$  be a  $\Pi_1^0$  subset of  $2^\omega$  of positive measure. Let  $\mathbf{p} \in \mathcal{P}_w$  be the weak degree of  $P$ . Then  $\mathbf{p} < \mathbf{1}$ . The same holds for strong degrees.*

*Proof.* This follows from Corollary 7.6 by setting  $\mathbf{q} = \mathbf{0}$ .  $\square$

**Remark 7.8.** In [48] we shall give an example of a  $\Pi_1^0$  set  $Q \subseteq 2^\omega$  whose Turing upward closure  $\widehat{Q} = \{f \in 2^\omega \mid (\exists g \leq_T f)(g \in Q)\}$  is of positive measure yet does not contain any  $\Pi_1^0$  set of positive measure.

**Definition 7.9.** Following Jockusch [25], for  $k \geq 2$  we define

$$\text{DNR}_k = \{f \in \omega^\omega \mid \forall n (f(n) < k \text{ and } f(n) \neq \{n\}(n))\}.$$

Thus  $\text{DNR}_k$  is the set of  $k$ -bounded, diagonally nonrecursive functions. Note that  $\text{DNR}_k$  is recursively bounded and  $\Pi_1^0$ .

**Remark 7.10.** By Jockusch [25, Theorem 5] each  $\text{DNR}_k$  is weakly complete, i.e., of weak degree  $\mathbf{1}$ . Let  $\mathbf{d}_k^* \in \mathcal{P}_s$  be the strong degree of  $\text{DNR}_k$ . It is well known (see also the proof of Corollary 7.6 above) that  $\text{DNR}_2$  is strongly complete, i.e.,  $\mathbf{d}_2^* = \mathbf{1}$  in  $\mathcal{P}_s$ . By Jockusch [25, Theorem 6] we have

$$\mathbf{1} = \mathbf{d}_2^* > \mathbf{d}_3^* > \cdots > \mathbf{d}_k^* > \mathbf{d}_{k+1}^* > \cdots$$

in  $\mathcal{P}_s$ . See also Simpson [54, Remark 3.21].

We have the following new result.

**Corollary 7.11.** *Let  $P$  and  $Q$  be  $\Pi_1^0$  subsets of  $2^\omega$ . Assume that  $P$  is of positive measure. Let  $\mathbf{p}, \mathbf{q} \in \mathcal{P}_s$  be the strong degrees of  $P, Q$  respectively. For each  $k \geq 2$ , if  $\mathbf{d}_k^* \leq \sup(\mathbf{p}, \mathbf{q})$ , then  $\mathbf{d}_k^* \leq \mathbf{q}$ . In particular we have*

$$\mathbf{1} = \sup(\mathbf{p}, \mathbf{d}_2^*) > \sup(\mathbf{p}, \mathbf{d}_3^*) > \cdots > \sup(\mathbf{p}, \mathbf{d}_k^*) > \sup(\mathbf{p}, \mathbf{d}_{k+1}^*) > \cdots.$$

*Proof.* It suffices to note that  $\text{DNR}_k$  is of the form required by Lemma 7.2. Namely,  $\text{DNR}_k = \prod_{n \in \omega} F_n$  where  $F_n = \{m < k \mid \{n\}(n) \neq m\}$ .  $\square$

**Remark 7.12.** In Corollary 7.11, we do not know whether it is necessarily the case that  $\mathbf{d}_3^* \geq \mathbf{p}$ , or  $\mathbf{d}_k^* \geq \mathbf{p}$  for all  $k \geq 3$ .

## 8 $\Pi_1^0$ sets of random reals

In this section we exhibit a particular degree  $\mathbf{r} \in \mathcal{P}_w$  and note some of its degree-theoretic properties.

As in Section 7, let  $\mu$  denote the fair coin probability measure on  $2^\omega$ .

**Definition 8.1.** An *effective null*  $G_\delta$  is a set  $S \subseteq 2^\omega$  of the form  $S = \bigcap_{n \in \omega} U_n$  where  $\{U_n\}_{n \in \omega}$  is a recursive sequence of  $\Sigma_1^0$  subsets of  $2^\omega$  with  $\mu(U_n) \leq 1/2^n$  for all  $n$ . A point  $f \in 2^\omega$  is said to be *random* if  $f \notin S$  for all effective null  $G_\delta$  sets  $S \subseteq 2^\omega$ .

**Remark 8.2.** The notion of randomness in Definition 8.1 is due to Martin-Löf [39] and has been studied extensively. It appears to be the most general and natural notion of algorithmic randomness for infinite sequences of 0's and 1's. It has also been called *Martin-Löf randomness* (Li/Vitányi [37, Section 2.5]), *1-randomness* (Kurtz [28], Kautz [27]), and *the NAP property* (Kučera [29, 30, 31, 32, 33, 34, 35]). It is closely related to Kolmogorov complexity (see Li/Vitányi [37]).

The following theorem is well known. It says that the union of all effective null  $G_\delta$  sets is an effective null  $G_\delta$  set.

**Theorem 8.3.**  $\{f \in 2^\omega \mid f \text{ is not random}\}$  is an effective null  $G_\delta$  set.

*Proof.* This result is essentially due to Martin-Löf [39]. See also Kučera [29, Theorems 1 and 2]. For the sake of completeness, we present the proof here. For each  $e \in \omega$  define a  $\Sigma_1^0$  set  $V_e \subseteq 2^\omega$  as follows. Compute  $\{e\}(e)$ . If  $\{e\}(e)$  is undefined,  $V_e =$  the empty set. If  $\{e\}(e) = m$ , let  $V_e =$  the  $\Sigma_1^0$  subset of  $2^\omega$  with  $\Sigma_1^0$  index  $m$  enumerated so long as its measure is  $\leq 1/2^e$ . Put  $R = \bigcup_{k=0}^\infty R_k$  where  $R_k = 2^\omega \setminus \bigcup_{e=k+1}^\infty V_e$ . We have

$$\mu(2^\omega \setminus R_k) \leq \sum_{e=k+1}^\infty \mu(V_e) \leq \sum_{e=k+1}^\infty \frac{1}{2^e} = \frac{1}{2^k}$$

and  $R_k$  is uniformly  $\Pi_1^0$ . Thus  $2^\omega \setminus R = \bigcap_{k=0}^\infty (2^\omega \setminus R_k)$  is an effective null  $G_\delta$  set. Hence every random  $f \in 2^\omega$  belongs to  $R$ . We claim that, conversely, every  $f \in R$  is random. To see this, consider an effective null  $G_\delta$  set  $S = \bigcap_{n=0}^\infty U_n$ , where  $\mu(U_n) \leq 1/2^n$ . It suffices to show that  $R \cap S = \emptyset$ . Given  $k \in \omega$ , let  $e \geq k+1$  be such that, for all  $n$ ,  $\{e\}(n)$  is a  $\Sigma_1^0$  index of  $U_n$ . In particular,  $\{e\}(e)$  is a  $\Sigma_1^0$  index of  $U_e$ . Since  $\mu(U_e) \leq 1/2^e$ , it follows that  $V_e = U_e$ . Since  $R_k$  is disjoint from  $V_e$ , it follows that  $R_k$  is disjoint from  $S$ . But  $k$  is arbitrary, so  $R$  is disjoint from  $S$ . This completes the proof that  $R = \{f \in 2^\omega \mid f \text{ is random}\}$ .  $\square$

**Corollary 8.4.** *There exists a nonempty  $\Pi_1^0$  set*

$$P \subseteq R = \{f \in 2^\omega \mid f \text{ is random}\}.$$

*Proof.* Trivially any effective null  $G_\delta$  set is  $\Pi_2^0$ . In particular, by Theorem 8.3,  $R$  is  $\Sigma_2^0$ . Hence  $R$  is a union of  $\Pi_1^0$  sets. Let  $P$  be any one of these  $\Pi_1^0$  sets. Alternatively, we could let  $P$  be any one of the sets  $R_k$  as in the proof of Theorem 8.3. Each of these sets is  $\Pi_1^0$ .  $\square$

**Notation 8.5.** We use the following notation for shifts:  $f^{(k)}(n) = f(k+n)$ . Note that  $f \mapsto f^{(k)}$  is a mapping of  $2^\omega$  into  $2^\omega$ .

**Lemma 8.6.** *For all  $f \in 2^\omega$  and  $k \in \omega$ ,  $f$  is random if and only if  $f^{(k)}$  is random.*

*Proof.* The proof is straightforward.  $\square$

The next lemma is an effective version of the Zero-One Law of probability theory.

**Lemma 8.7.** *Let  $f$  be random. Let  $P \subseteq 2^\omega$  be  $\Pi_1^0$  with  $\mu(P) > 0$ . Then  $\exists k (f^{(k)} \in P)$ .*

*Proof.* This is due to Kučera [29]. For completeness we present the proof here. Let  $P$  be the set of paths through  $T$ , where  $T \subseteq 2^{<\omega}$  is a recursive tree. Put

$$\tilde{T} = \{\sigma \frown \langle i \rangle \mid \sigma \in T, i \in \{0, 1\}, \sigma \frown \langle i \rangle \notin T\}.$$

For  $n \geq 1$ , put

$$T^n = \{\tau_1 \frown \cdots \frown \tau_m \frown \sigma \mid m < n, \tau_1, \dots, \tau_m \in \tilde{T}, \sigma \in T\},$$

and let  $P^n = [T^n]$ , the set of paths through  $T^n$ . We have

$$\mu(P^n) = 1 - (1 - \mu(P))^n,$$

hence  $2^\omega \setminus \bigcup_{n=1}^\infty P^n$  is an effective null  $G_\delta$  set. Hence  $f \in P^n$  for some  $n$ . Hence for some  $m < n$  and  $\tau_1, \dots, \tau_m \in \tilde{T}$  we have  $f = \tau_1 \frown \cdots \frown \tau_m \frown g$  where  $g \in P$ . Putting  $k = \text{length of } \tau_1 \frown \cdots \frown \tau_m$ , we have  $g = f^{(k)}$ .  $\square$

**Lemma 8.8.** *Let  $f \in 2^\omega$  be random. Then for all  $\Pi_1^0$  sets  $P \subseteq 2^\omega$ , if  $f \in P$  then  $\mu(P) > 0$ .*

*Proof.* Since  $P$  is  $\Pi_1^0$ , let  $P = \bigcap_s P_s$  where  $P_s$ ,  $s \in \omega$ , is a recursive sequence of canonically indexed clopen sets in  $2^\omega$  with

$$P_0 \supseteq P_1 \supseteq \cdots \supseteq P_s \supseteq P_{s+1} \supseteq \cdots.$$

Assume  $\mu(P) = 0$ . By countable additivity,  $\lim_s \mu(P_s) = 0$ . Define a recursive function  $h : \omega \rightarrow \omega$  by  $h(n) = \text{least } s \text{ such that } \mu(P_s) \leq 1/2^n$ . Putting  $U_n = P_{h(n)}$ , we see that  $P = \bigcap_n U_n$  is an effective null  $G_\delta$  set. Hence  $f \notin P$ , a contradiction.  $\square$

**Lemma 8.9.** *Let  $P$  and  $R$  be as in Corollary 8.4. Then  $\mu(P) > 0$ , and  $P \equiv_w R$ .*

*Proof.* By Lemma 8.8  $\mu(P) > 0$ , and by Lemma 8.7  $(\forall f \in R) \exists k (f^{(k)} \in P)$ . Thus  $P \leq_w R$ . On the other hand, since  $P \subseteq R$ ,  $P \geq_w R$ , so  $P \equiv_w R$ .  $\square$

**Theorem 8.10.** *Let  $\mathbf{r} = \text{deg}_w(R)$  where  $R = \{f \in 2^\omega \mid f \text{ is random}\}$ . Then  $\mathbf{r}$  can be characterized as the unique largest weak degree of a  $\Pi_1^0$  set  $P \subseteq 2^\omega$  such that  $\mu(P) > 0$ .*

*Proof.* By Lemma 8.7 we have that, for any  $\Pi_1^0$  set  $P \subseteq 2^\omega$  with  $\mu(P) > 0$ ,  $P \leq_w R$ . By Corollary 8.4 let  $P'$  be a nonempty  $\Pi_1^0$  subset of  $R$ . By Lemma 8.9 we have  $\mu(P') > 0$  and  $P' \equiv_w R$ . This completes the proof.  $\square$

**Remark 8.11.** Theorem 8.10 tells us that, among all weak degrees of  $\Pi_1^0$  sets of positive measure, there exists a unique largest degree. Simpson/Slaman [55] and independently Terwijn [58] have shown that, among all strong degrees of  $\Pi_1^0$  sets of positive measure, there is no largest or even maximal degree.

We end this section by noting some additional properties of the particular weak degree  $\mathbf{r}$  which was defined in Theorem 8.10.

**Theorem 8.12.** *Let  $\mathbf{r}$  be the weak degree of  $R = \{f \in 2^\omega \mid f \text{ is random}\}$ . Then:*

1.  $\mathbf{r} \in \mathcal{P}_w$ , and  $\mathbf{0} < \mathbf{r} < \mathbf{1}$ .
2. For all  $\mathbf{q} \in \mathcal{P}_w$ , if  $\mathbf{q} < \mathbf{1}$  then  $\sup(\mathbf{q}, \mathbf{r}) < \mathbf{1}$ .
3. For all  $\mathbf{q}_1, \mathbf{q}_2 \in \mathcal{P}_w$ , if  $\mathbf{r} \geq \inf(\mathbf{q}_1, \mathbf{q}_2)$  then either  $\mathbf{r} \geq \mathbf{q}_1$  or  $\mathbf{r} \geq \mathbf{q}_2$ .
4. There is no separating  $\mathbf{s} \in \mathcal{P}_w$  such that  $\mathbf{0} < \mathbf{s} \leq \mathbf{r}$ .

*Proof.* Since  $R$  has no recursive members,  $\mathbf{r} > \mathbf{0}$ . Theorem 8.10 implies that  $\mathbf{r} \in \mathcal{P}_w$  and contains a  $\Pi_1^0$  subset of  $2^\omega$  of positive measure. By Corollary 7.7 it follows that  $\mathbf{r} < \mathbf{1}$ , completing the proof of part 1 of the theorem. Corollary 7.6 gives part 2. To prove part 3, let  $Q_1, Q_2$  be  $\Pi_1^0$  subsets of  $2^\omega$  of weak degree  $\mathbf{q}_1, \mathbf{q}_2$  respectively. We are assuming that  $R \geq_w Q_1 \cup Q_2$ . By Corollary 8.4 let  $P$  be a nonempty  $\Pi_1^0$  subset of  $R$ . Then  $P \geq_w Q_1 \cup Q_2$ . By Lemma 6.9 we can find a nonempty  $\Pi_1^0$  set  $\overline{P} \subseteq P$  such that  $\overline{P} \geq_s Q_1 \cup Q_2$ . Thus there is a recursive

functional  $\Phi : \overline{P} \rightarrow Q_1 \cup Q_2$ . Put  $\overline{P}_1 = \overline{P} \cap \Phi^{-1}(Q_1)$  and  $\overline{P}_2 = \overline{P} \cap \Phi^{-1}(Q_2)$ . We have  $\overline{P} = \overline{P}_1 \cup \overline{P}_2$ , hence at least one of  $\overline{P}_1$  and  $\overline{P}_2$  is nonempty, say  $\overline{P}_1$ . Then  $\overline{P}_1 \geq_s Q_1$ . Note also that  $\overline{P}_1$  is a  $\Pi_1^0$  subset of  $R$ , hence by Lemma 8.9 we have  $\overline{P}_1 \equiv_w R$ . It follows that  $R \geq_w Q_1$ , and this proves part 3. Part 4 is a consequence of Theorem 7.5.  $\square$

**Corollary 8.13.** *The weak degree  $\mathbf{r} \in \mathcal{P}_w$  is meet irreducible and does not join to  $\mathbf{1}$  in  $\mathcal{P}_w$ .*

*Proof.* This follows from parts 1, 2 and 3 of Theorem 8.12.  $\square$

## 9 Thin $\Pi_1^0$ subsets of $2^\omega$

In this section we discuss an interesting class of degrees in  $\mathcal{P}_w$ , each of which is incomparable with the particular degree  $\mathbf{r} \in \mathcal{P}_w$  of Section 8.

We begin with some generalities concerning thin  $\Pi_1^0$  sets.

**Definition 9.1.** A  $\Pi_1^0$  set  $Q \subseteq \omega^\omega$  is said to be *thin* if, for all  $\Pi_1^0$  sets  $Q' \subseteq Q$ , the set-theoretic difference  $Q \setminus Q'$  is  $\Pi_1^0$ .

**Lemma 9.2.** *Let  $Q$  be a recursively bounded  $\Pi_1^0$  set. Then  $Q$  is thin if and only if all  $\Pi_1^0$  subsets of  $Q$  are trivial, i.e., they are of the form*

$$Q' = \{g \in Q \mid \sigma_1 \subset g \text{ or } \cdots \text{ or } \sigma_k \subset g\}$$

for some finite set of strings  $\sigma_1, \dots, \sigma_k$ .

*Proof.* If  $Q$  is any  $\Pi_1^0$  set, and if  $Q' \subseteq Q$  is trivial, then clearly  $Q'$  and  $Q \setminus Q'$  are  $\Pi_1^0$ . It remains to prove that if  $Q$  is a recursively bounded  $\Pi_1^0$  set, and if  $Q' \subseteq Q$  is such that  $Q'$  and  $Q \setminus Q'$  are  $\Pi_1^0$ , then  $Q'$  is trivial. To see this, let  $h$  be a recursive function such that  $\forall n (\forall g \in Q) (g(n) < h(n))$ . As in the proof of Theorem 4.7, note that  $Q$  is a closed set in the compact space  $Q_h = \{g \mid \forall n (g(n) < h(n))\}$ . By Theorem 4.3, since  $Q'$  and  $Q \setminus Q'$  are  $\Pi_1^0$ , let  $T'$  and  $T''$  be recursive trees such that  $Q' = [T']$  and  $Q \setminus Q' = [T'']$ . Then  $Q_h$  is covered by clopen sets of the form  $\{g \mid \tau \subset g\}$  where either  $\tau \notin T'$  or  $\tau \notin T''$ . Let  $\tau_1, \dots, \tau_l$  be a finite subcovering. Let  $\sigma_1, \dots, \sigma_k$  consist of those  $\tau_i$ ,  $1 \leq i \leq l$ , such that  $\tau_i \notin T''$ . Then  $Q' = \{g \in Q \mid \sigma_1 \subset g \text{ or } \cdots \text{ or } \sigma_k \subset g\}$ , so  $Q'$  is trivial.  $\square$

**Theorem 9.3.** *Let  $Q$  be a nonempty thin recursively bounded  $\Pi_1^0$  set. Then  $f \in Q$  is isolated if and only if  $f$  is recursive. In particular,  $Q$  is perfect if and only if and only if  $Q$  has no recursive members, i.e.,  $\deg_w(Q) > \mathbf{0}$ .*

*Proof.* If  $f \in Q$  is isolated, then  $f$  is recursive by Theorem 4.15. Conversely, suppose  $f \in Q$  is recursive. Then the singleton set  $\{f\}$  is a  $\Pi_1^0$  subset of  $Q$ . Since  $Q$  is thin and recursively bounded  $\Pi_1^0$ , by Lemma 9.2 there is a finite set of strings  $\sigma_1, \dots, \sigma_k$  such that  $f$  is the unique  $g \in Q$  such that  $\sigma_1 \subset g \text{ or } \cdots \text{ or } \sigma_k \subset g$ . It follows that  $f$  is isolated. This proves the first part of the theorem. The second part follows immediately.  $\square$

**Lemma 9.4.** *Let  $Q$  be a thin recursively bounded  $\Pi_1^0$  set. Then:*

1. *Every  $\Pi_1^0$  subset of  $Q$  is thin and recursively bounded.*
2. *Let  $P = \{\Phi(g) \mid g \in Q\}$  be the image of  $Q$  under a recursive functional  $\Phi : Q \rightarrow \omega^\omega$ . Then  $P$  is a thin recursively bounded  $\Pi_1^0$  set.*

*Proof.* Part 1 is straightforward. For part 2, note first that, since  $Q$  is recursively bounded and  $\Pi_1^0$ , so is  $P$ , by Theorem 4.7. It remains to show that  $P$  is thin. Given a  $\Pi_1^0$  set  $P' \subseteq P$ , let  $Q' = \{g \in Q \mid \Phi(g) \in P'\}$  be the preimage of  $P'$ . By Theorem 4.4,  $Q'$  is  $\Pi_1^0$ . Since  $Q$  is thin,  $Q \setminus Q'$  is also  $\Pi_1^0$ . It follows by Theorem 4.7 that  $P \setminus P' = \{\Phi(g) \mid g \in Q \setminus Q'\}$  is  $\Pi_1^0$ . Since  $P'$  is an arbitrary  $\Pi_1^0$  subset of  $P$ , we see that  $P$  is thin.  $\square$

**Theorem 9.5.** *If  $Q$  is a thin recursively bounded  $\Pi_1^0$  set, then  $Q$  is recursively homeomorphic to a thin  $\Pi_1^0$  set  $Q^* \subseteq 2^\omega$ . Moreover,  $Q$  is perfect if and only if  $Q^*$  is perfect.*

*Proof.* This follows from Theorems 4.10 and 9.3 and part 2 of Lemma 9.4.  $\square$

**Remark 9.6.** There is a large literature on thin perfect  $\Pi_1^0$  subsets of  $2^\omega$  going back to Martin/Pour-El [38]. See Downey/Jockusch/Stob [15, 16] and Cholak et al [11]. Typically, thin perfect  $\Pi_1^0$  subsets of  $2^\omega$  are constructed by means of priority arguments. In this sense, thin perfect  $\Pi_1^0$  subsets of  $2^\omega$  and their weak and strong degrees are artificial or unnatural. In particular, thin perfect  $\Pi_1^0$  subsets of  $2^\omega$  have been used by Binns/Simpson [6] to embed countable distributive lattices into  $\mathcal{P}_w$  and  $\mathcal{P}_s$ .

**Remark 9.7.** Let  $\mathbf{q} = \deg_w(Q)$  where  $Q$  is any nonempty thin perfect recursively bounded  $\Pi_1^0$  set. Obviously  $\mathbf{q} \in \mathcal{P}_w$ . Let  $\mathbf{r} = \deg_w(R)$  where  $R = \{f \in 2^\omega \mid f \text{ is random}\}$ . We have seen in Theorem 8.10 that  $\mathbf{r} \in \mathcal{P}_w$ . Our goal in this section is to prove Theorem 9.15, which says that  $\mathbf{q}$  and  $\mathbf{r}$  are incomparable, i.e.,  $\mathbf{q} \not\leq \mathbf{r}$  and  $\mathbf{r} \not\leq \mathbf{q}$ . By Theorem 9.5, it suffices to prove this in the special case when  $Q$  is a nonempty thin perfect  $\Pi_1^0$  subset of  $2^\omega$ .

**Lemma 9.8.** *Let  $Q$  be a thin  $\Pi_1^0$  subset of  $2^\omega$ . Then  $\mu(Q) = 0$ .*

*Proof.* For  $f, g \in 2^\omega$  we write  $f <_{\text{lex}} g$  to mean that there exists  $j$  such that  $(\forall i < j) (f(i) = g(i))$  and  $f(j) < g(j)$ . Note that  $<_{\text{lex}}$  is a linear ordering of  $2^\omega$ , the *lexicographical ordering*. For  $\sigma, \tau \in 2^{<\omega}$  we write  $\sigma <_{\text{lex}} \tau$  to mean that there exists  $j < \min(\text{lh}(\sigma), \text{lh}(\tau))$  such that  $(\forall i < j) (\sigma(i) = \tau(i))$  and  $\sigma(j) < \tau(j)$ . Note that, for each  $n \in \omega$ , the restriction of  $<_{\text{lex}}$  to strings of length  $n$  is a linear ordering, the *lexicographical ordering*.

Let  $Q$  be a thin  $\Pi_1^0$  subset of  $2^\omega$ . Assume for a contradiction that  $\mu(Q) > 0$ . Fix  $p \in \omega$  such that  $\mu(Q) > 1/2^p$ . Put  $T = \text{Ext}(Q) = \{\tau \in 2^{<\omega} \mid (\exists g \in Q) (\tau \subset g)\}$ , the set of extendible nodes of  $Q$ . By Lemma 4.13  $T$  is a  $\Pi_1^0$  tree, and  $Q = [T]$ . Define inductively a sequence of strings  $\tau_n \in T$ ,  $n \in \omega$ , as follows:  $\tau_n$  = the lexicographically least  $\tau \in T$  of length  $p + n + 1$  such that  $\tau >_{\text{lex}} \tau_m$

for all  $m < n$ . The existence of  $\tau_n$  is assured by the fact that  $\{g \in Q \mid \tau_0 \subset g \text{ or } \dots \text{ or } \tau_{n-1} \subset g\}$  is a lexicographically initial segment of  $Q$  of measure

$$\sum_{m=0}^{n-1} \mu(\{g \in Q \mid \tau_m \subset g\}) \leq \sum_{m=0}^{n-1} \frac{1}{2^{p+m+1}} < \frac{1}{2^p} < \mu(Q).$$

Put  $Q' = \{g \in Q \mid \neg \exists n (\tau_n \subset g)\}$ . Thus  $Q'$  is a lexicographically final segment of  $Q$ . Moreover  $Q \setminus Q' = \{g \in Q \mid \exists n (\tau_n \subset g)\}$  is clearly not compact, hence not of the form  $\{g \in Q \mid \sigma_1 \subset g \text{ or } \dots \text{ or } \sigma_k \subset g\}$  where  $\sigma_1, \dots, \sigma_k$  is a finite set of strings. It follows that  $Q'$  is also not of this form. In the next paragraph we shall show that  $Q'$  is a  $\Pi_1^0$  subset of  $Q$ . Hence  $Q$  is not thin, so our lemma will be proved.

It remains to show that  $Q'$  is  $\Pi_1^0$ . Since  $T$  is  $\Pi_1^0$ , let  $T^s$ ,  $s \in \omega$  be a recursive sequence of recursive trees such that

$$T^0 \supseteq T^1 \supseteq \dots \supseteq T^s \supseteq T^{s+1} \supseteq \dots$$

and  $\bigcap_{s=0}^{\infty} T^s = T$ . For each  $s$  define inductively  $\tau_n^s =$  the lexicographically least  $\tau \in T^s$  of length  $p + n + 1$  such that  $\tau >_{\text{lex}} \tau_m^s$  for all  $m < n$ . The double sequence  $\tau_n^s$ ,  $n, s \in \omega$  is recursive, and for each  $n$  we have

$$\tau_n^0 \leq_{\text{lex}} \tau_n^1 \leq_{\text{lex}} \dots \leq_{\text{lex}} \tau_n^s \leq_{\text{lex}} \tau_n^{s+1} \leq_{\text{lex}} \dots$$

and  $\tau_n = \lim_s \tau_n^s$ . Since  $Q'$  is a lexicographically final segment of  $Q = [T]$ , it follows that

$$Q' = \{g \in Q \mid \neg \exists n (\tau_n \subset g)\} = \{g \in Q \mid \neg \exists s \exists n (\tau_n^s \subset g)\}.$$

Thus  $Q'$  is  $\Pi_1^0$ , and our lemma is proved.  $\square$

**Remark 9.9.** We do not know the answer to the following question. If  $Q$  is a thin perfect  $\Pi_1^0$  subset of  $2^\omega$ , does it follow that the Turing upward closure  $\widehat{Q} = \{f \in 2^\omega \mid (\exists g \leq_T f) (g \in Q)\}$  is of measure 0?

**Lemma 9.10.** *Let  $Q$  be a nonempty thin  $\Pi_1^0$  subset of  $2^\omega$ . If  $f$  is random, and if  $g \in Q$  is almost recursive, then  $f \not\leq_T g$ . In particular,  $R \not\leq_w Q$ .*

*Proof.* Assume for a contradiction that  $f \in R$  and  $f \leq_T g$ . Since  $g$  is almost recursive, by Theorem 4.18 we have  $f \leq_{tt} g$ , i.e.,  $f = \Phi(g)$  where  $\Phi : 2^\omega \rightarrow 2^\omega$  is a total recursive functional. Since  $R$  is  $\Sigma_2^0$ , let  $P$  be a  $\Pi_1^0$  subset of  $R$  such that  $f \in P$ . Put  $\overline{Q} = \{\overline{g} \in Q \mid \Phi(\overline{g}) \in P\}$ . Then  $\overline{Q}$  is a  $\Pi_1^0$  subset of  $Q$ . By part 1 of Lemma 9.4,  $\overline{Q}$  is thin. Note also that  $g \in \overline{Q}$ , since  $\Phi(g) = f \in P$ . Put  $\overline{P} = \{\Phi(\overline{g}) \mid \overline{g} \in \overline{Q}\}$ . Clearly  $\overline{P} \subseteq P \subseteq R$ , and by Theorem 4.7 we have that  $\overline{P}$  is  $\Pi_1^0$ . Moreover  $f = \Phi(g) \in \overline{P}$ , so  $\overline{P}$  is nonempty, so by Lemma 8.8 we have  $\mu(\overline{P}) > 0$ . On the other hand, by part 2 of Lemma 9.4,  $\overline{P}$  is thin, hence by Lemma 9.8 we have  $\mu(\overline{P}) = 0$ . This contradiction proves the first part of our lemma. For the second part, since  $Q$  is nonempty, the Almost Recursive Basis Theorem 4.19 gives  $g \in Q$  such that  $g$  is almost recursive. By the first part of the lemma,  $f \not\leq_T g$  for all  $f \in R$ . It follows that  $R \not\leq_w Q$ .  $\square$

**Lemma 9.11.** *Let  $f \in 2^\omega$  be random. If  $g \leq_{tt} f$  is nonrecursive, then there exists  $\bar{f} \in 2^\omega$  such that  $\bar{f} \equiv_T g$  and  $\bar{f}$  is random.*

*Proof.* This lemma has been stated by Demuth [14, Lemma 30]. The proof is in Kautz's thesis [27, Theorem IV.3.16].  $\square$

**Lemma 9.12.** *Let  $f \in 2^\omega$  be random and almost recursive. If  $g \leq_T f$  is nonrecursive, then there exists  $\bar{f} \in 2^\omega$  such that  $\bar{f} \equiv_T g$  and  $\bar{f}$  is random.*

*Proof.* This follows from Lemma 9.11 because, by Theorem 4.18, if  $f$  is almost recursive then  $g \leq_T f$  implies  $g \leq_{tt} f$ .  $\square$

**Lemma 9.13.** *Let  $Q$  be a nonempty thin perfect  $\Pi_1^0$  subset of  $2^\omega$ . If  $g \in Q$ , and if  $f \in 2^\omega$  is random and almost recursive, then  $g \not\leq_T f$ . In particular  $Q \not\leq_w R$ , and  $Q \not\leq_w P$  for all  $\Pi_1^0$  sets  $P \subseteq 2^\omega$  of positive measure.*

*Proof.* Assume for a contradiction that  $g \in Q$  and  $g \leq_T f$ . Since  $f$  is almost recursive,  $g$  is almost recursive. Since  $f$  is random and almost recursive, Lemma 9.12 gives us  $\bar{f} \equiv_T g$  such that  $\bar{f}$  is random. This contradicts Lemma 9.10. The first part of our lemma is now proved. For the second part, let  $P \subseteq 2^\omega$  be  $\Pi_1^0$  of positive measure. Since  $R$  is  $\Sigma_2^0$  of measure 1, we can find a  $\Pi_1^0$  set  $P' \subseteq P \cap R$  which is of positive measure. By the Almost Recursive Basis Theorem 4.19, let  $f \in P'$  be almost recursive. By the first part of our lemma, we have  $g \not\leq_T f$  for all  $g \in Q$ . Thus  $Q \not\leq_w P'$ . It follows that  $Q \not\leq_w P$  and  $Q \not\leq_w R$ .  $\square$

Summarizing, we have:

**Lemma 9.14.** *Let  $f, g \in 2^\omega$  be almost recursive. Assume that  $f$  is random, and assume that  $g \in Q$  where  $Q$  is a thin perfect  $\Pi_1^0$  subset of  $2^\omega$ . Then  $f \not\leq_T g$  and  $g \not\leq_T f$ , i.e., the Turing degrees of  $f$  and  $g$  are incomparable.*

*Proof.* This is immediate from Lemmas 9.10 and 9.13.  $\square$

**Theorem 9.15.** *Let  $\mathbf{q} = \deg_w(Q)$  where  $Q$  is a nonempty thin perfect  $\Pi_1^0$  subset of  $2^\omega$ . Let  $\mathbf{r} = \deg_w(R)$  where  $R = \{f \in 2^\omega \mid f \text{ is random}\}$ . Then  $\mathbf{q}$  and  $\mathbf{r}$  are incomparable weak degrees in  $\mathcal{P}_w$ .*

*Proof.* Obviously  $\mathbf{q} \in \mathcal{P}_w$ . Theorem 8.10 implies that  $\mathbf{r} \in \mathcal{P}_w$ . By Lemma 9.10 we have  $\mathbf{r} \not\leq \mathbf{q}$ . By Lemma 9.13 we have  $\mathbf{q} \not\leq \mathbf{r}$ .  $\square$

**Corollary 9.16.** *There exist  $0 < \mathbf{q} < \mathbf{q}^*$  in  $\mathcal{P}_w$  such that  $\mathbf{q}$  is separating and  $\mathbf{q}^*$  is not separating. Indeed, every separating  $\mathbf{s} \in \mathcal{P}_w$  which is  $\leq \mathbf{q}^*$  is  $\leq \mathbf{q}$ .*

*Proof.* By Martin/Pour-El [38] let  $Q \subseteq 2^\omega$  be a thin perfect  $\Pi_1^0$  set which is separating. Put  $\mathbf{q} = \deg_w(Q)$  and  $\mathbf{q}^* = \sup(\mathbf{q}, \mathbf{r})$ . By Theorem 9.15 we have  $\mathbf{q} < \mathbf{q}^*$ . If  $\mathbf{s}$  were separating and  $\leq \mathbf{q}^*$  but not  $\leq \mathbf{q}$ , then this would contradict Theorem 7.5.  $\square$

## 10 Some additional natural examples in $\mathcal{P}_w$

In this section we present some additional natural examples in  $\mathcal{P}_w$ , including a hierarchy of weak degrees in  $\mathcal{P}_w$  corresponding to the transfinite Ackermann hierarchy from proof theory.

**Definition 10.1.** Put  $\text{DNR} = \{f \in \omega^\omega \mid \forall n (f(n) \neq \{n\}(n))\}$ , the set of *diagonally nonrecursive* functions. The set of Turing degrees of members of DNR has been studied by Jockusch [25]. Note that DNR is nonempty and  $\Pi_1^0$ . If  $h : \omega \rightarrow \omega$  is a recursive function such that  $h(n) \geq 2$  for all  $n$ , put  $\text{DNR}_h = \{f \in \text{DNR} \mid \forall n (f(n) < h(n))\}$ , the set of  $h$ -bounded DNR functions. In addition, put  $\text{DNR}_{\text{REC}} = \bigcup \{\text{DNR}_h \mid h \text{ is recursive}\}$ , the set of recursively bounded DNR functions.

**Remark 10.2.** Trivially

$$\text{DNR} \supset \text{DNR}_{\text{REC}} \supset \text{DNR}_h,$$

hence  $\text{DNR} \leq_s \text{DNR}_{\text{REC}} \leq_s \text{DNR}_h$ , hence  $\text{DNR} \leq_w \text{DNR}_{\text{REC}} \leq_w \text{DNR}_h$ . According to Ambos-Spies et al [2, Theorems 1.8 and 1.9], we have strict inequalities

$$\text{DNR} <_w \text{DNR}_{\text{REC}} <_w \text{DNR}_h.$$

As in Section 8, let  $R$  be the set of random reals. An argument of Kurtz (see Jockusch [25, Proposition 3]) shows that  $\text{DNR}_h \leq_w R$  provided  $h$  is such that  $\sum_{n=0}^{\infty} 1/h(n) < \infty$ , for example  $h(n) = \max(n^2, 2)$ .

**Remark 10.3.** Since  $\text{DNR}_h$  is nonempty, recursively bounded, and  $\Pi_1^0$ , we have  $\text{deg}_s(\text{DNR}_h) \in \mathcal{P}_s$  and  $\text{deg}_w(\text{DNR}_h) \in \mathcal{P}_w$ . Although DNR and  $\text{DNR}_{\text{REC}}$  are not recursively bounded, it will be shown in Simpson [48] that  $\text{deg}_w(\text{DNR}) \in \mathcal{P}_w$  and  $\text{deg}_w(\text{DNR}_{\text{REC}}) \in \mathcal{P}_w$ . We do not know whether  $\text{deg}_s(\text{DNR}) \in \mathcal{P}_s$ , or whether  $\text{deg}_s(\text{DNR}_{\text{REC}}) \in \mathcal{P}_s$ . Put  $\mathbf{d} = \text{deg}_w(\text{DNR})$ ,  $\mathbf{d}_{\text{REC}} = \text{deg}_w(\text{DNR}_{\text{REC}})$ ,  $\mathbf{d}_h = \text{deg}_w(\text{DNR}_h)$ ,  $\mathbf{r} = \text{deg}_w(R)$ . Summarizing, we have the following result.

**Theorem 10.4.** *In  $\mathcal{P}_w$  we have*

$$\mathbf{0} < \mathbf{d} < \mathbf{d}_{\text{REC}} < \mathbf{d}_h < \mathbf{r} < \mathbf{1}$$

for all sufficiently fast-growing recursive functions  $h : \omega \rightarrow \omega$ .

*Proof.* This follows from part 1 of Theorem 8.12 plus the results of Ambos-Spies et al [2] and Simpson [48] which were mentioned in Remarks 10.2 and 10.3 above.  $\square$

**Remark 10.5.** Some of our natural weak degrees are closely related to certain formal systems which arise naturally in the foundations of mathematics. Namely, the weak degrees  $\mathbf{1}$ ,  $\mathbf{r}$ ,  $\mathbf{d}$  correspond to the systems  $\text{WKL}_0$ ,  $\text{WWKL}_0$ ,  $\text{RCA}_0 + \text{DNR}$  respectively. Each of these subsystems of second order arithmetic is of interest in connection with the well known foundational program of reverse mathematics. See Simpson [52, Chapter IV and Section X.1], Yu/Simpson [61], Brown/Giusto/Simpson [7], and Giusto/Simpson [23]. The standard reference for reverse mathematics is Simpson [52].

**Remark 10.6.** From the recursion-theoretic viewpoint, there are some subtle issues concerning naturalness of the mass problems  $\text{DNR}$ ,  $\text{DNR}_{\text{REC}}$ ,  $\text{DNR}_h$  and of their weak degrees  $\mathbf{d}$ ,  $\mathbf{d}_{\text{REC}}$ ,  $\mathbf{d}_h$ . First,  $\text{DNR}$ ,  $\text{DNR}_{\text{REC}}$ ,  $\text{DNR}_h$  are not invariant under recursive permutations of  $\omega$ , and on this basis it is possible to question their recursion-theoretic naturalness. (See also the discussion of the recursion-theoretic Erlanger Programm in Rogers [45, Chapter 4].) On the other hand, this objection clearly does not apply to the weak degrees  $\mathbf{d}$ ,  $\mathbf{d}_{\text{REC}}$ ,  $\mathbf{d}_h$ , because all weak and strong degrees are invariant under recursive permutations of  $\omega$ . Second, one may note that our definitions of  $\text{DNR}$ ,  $\text{DNR}_{\text{REC}}$ ,  $\text{DNR}_h$  and their weak degrees  $\mathbf{d}$ ,  $\mathbf{d}_{\text{REC}}$ ,  $\mathbf{d}_h$  depend upon a particular choice of Gödel numbering of Turing machines, because the function  $n \mapsto \{n\}(n)$  is defined in terms of such a Gödel numbering. (See also the discussion of acceptable Gödel numberings in Rogers [45].) We shall now present a method of overcoming this objection. Our idea is to replace the particular partial recursive function  $n \mapsto \{n\}(n)$  by an arbitrary partial recursive function  $n \mapsto \psi(n)$ . This will answer the objection, because the extensional concept “partial recursive function” is independent of the choice of Gödel numbering.

**Definition 10.7.** Let  $D$  be the set of  $g \in \omega^\omega$  such that for all partial recursive functions  $\psi$  there exists  $f \leq_T g$  such that  $\forall n (f(n) \neq \psi(n))$ . Let  $D_{\text{REC}}$  be the set of  $g \in \omega^\omega$  such that for all partial recursive functions  $\psi$  there exists  $f \leq_T g$  such that  $\forall n (f(n) < h(n) \text{ and } f(n) \neq \psi(n))$  for some recursive function  $h : \omega \rightarrow \omega$ .

**Remark 10.8.** Using the S-m-n Theorem, it is easy to see that  $\text{DNR} \equiv_w D$  and  $\text{DNR}_{\text{REC}} \equiv_w D_{\text{REC}}$ . (See also the proof of Theorem 10.10 below.) Thus the weak degrees  $\mathbf{d}$  and  $\mathbf{d}_{\text{REC}}$  are natural in the sense that they can be defined in a way that does not depend on the choice of Gödel numbering. What about  $\mathbf{d}_h$  where  $h$  is a fixed recursive function? Let  $D_h$  be the set of  $g \in \omega^\omega$  such that for all partial recursive functions  $\psi$  there exists  $f \leq_T g$  such that  $\forall n (f(n) < h(n) \text{ and } f(n) \neq \psi(n))$ . It is not clear that  $\text{DNR}_h \equiv_w D_h$  for a fixed recursive function  $h$ , but we have the following definition and theorem for classes of recursive functions.

**Definition 10.9.** If  $C$  is a class of recursive functions, put  $\text{DNR}_C = \bigcup_{h \in C} \text{DNR}_h$ . Let  $D_C$  be the set of  $g \in \omega^\omega$  such that for all partial recursive functions  $\psi$  there exists  $f \leq_T g$  such that  $\forall n (f(n) < h(n) \text{ and } f(n) \neq \psi(n))$  for some  $h \in C$ .

**Theorem 10.10.** *If  $C$  is closed under composition with primitive recursive functions, then  $\text{DNR}_C \equiv_w D_C$ . If there exists a uniform recursive enumeration of  $C$ , then  $\text{deg}_w(\text{DNR}_C) \in \mathcal{P}_w$ .*

*Proof.* Given  $g \in D_C$ , let  $f \leq_T g$  be as in the definition of  $D_C$  for the particular partial recursive function  $\psi(n) \simeq \{n\}(n)$ . Clearly  $f \in \text{DNR}_C$ , and this shows that  $\text{DNR}_C \leq_w D_C$ . Conversely, to show that  $\text{DNR}_C \geq_w D_C$ , let  $f \in \text{DNR}_C$  be given, say  $f \in \text{DNR}_h$  where  $h \in C$ . Given a partial recursive function  $\psi$ , apply the S-m-n Theorem to get a primitive recursive function  $p : \omega \rightarrow \omega$  such that  $\{p(n)\}(p(n)) \simeq \psi(n)$  for all  $n$ . Then we have  $\forall n (f(p(n)) < h(p(n)) \text{ and } f(p(n)) \neq \psi(n))$  and

$f(p(n)) \neq \psi(n)$ ). Moreover, the function  $n \mapsto f(p(n))$  is Turing reducible to  $f$ , and the function  $n \mapsto h(p(n))$  belongs to  $C$ . Thus  $f \in D_C$ . We have now shown that  $\text{DNR}_C \subseteq D_C$ . It follows that  $\text{DNR}_C \geq_w D_C$ , and we have proved the first part of the theorem. To prove the second part, let  $h_n, n \in \omega$  be a uniform recursive enumeration of  $C$ . Putting  $P_n = \text{DNR}_{h_n}$ , we note that  $P_n$  is uniformly recursively bounded and  $\Pi_1^0$ . As in the proof of Theorem 4.10, put  $P_n^* = \{G_f \mid f \in P_n\}$ . Thus  $P_n^*$  is a uniformly  $\Pi_1^0$  subset of  $2^\omega$  which is uniformly recursively homeomorphic to  $P_n$ . Put  $S^* = \bigcup_n P_n^*$ . Then  $S^*$  is a  $\Sigma_2^0$  subset of  $2^\omega$ , and by construction  $S^*$  is Turing degree isomorphic to  $\bigcup_n P_n = \text{DNR}_C$ . Now apply Theorem 10.11 to find a  $\Pi_1^0$  set  $P^* \subseteq 2^\omega$  such that  $P^*$  is Turing degree isomorphic to  $S^*$ . Then  $\text{deg}_w(\text{DNR}_C) = \text{deg}_w(S^*) = \text{deg}_w(P^*) \in \mathcal{P}_w$ .  $\square$

**Theorem 10.11.** *Let  $S$  be a  $\Sigma_2^0$  subset of  $2^\omega$ . Then we can find a  $\Pi_1^0$  set  $P \subseteq 2^\omega$  such that  $P$  is Turing degree isomorphic to  $S$ .*

*Proof.* We may safely assume that  $S$  is nonempty. By hypothesis  $S = \bigcup_n P_n$  where  $P_n, n \in \omega$  is a recursive sequence of nonempty  $\Pi_1^0$  subsets of  $2^\omega$ . We use a construction from Binns/Simpson [6, Definition 4.2]. Let  $T_n, n \in \omega$  be a recursive sequence of infinite recursive subtrees of  $2^{<\omega}$  such that  $P_n = [T_n]$ , the set of paths through  $T_n$ . Put

$$\tilde{T}_0 = \{\sigma \hat{\ } \langle i \rangle \mid \sigma \in T_0, i \in \{0, 1\}, \sigma \hat{\ } \langle i \rangle \notin T_0\}.$$

We may safely assume that  $\tilde{T}_0$  is infinite. Note that the strings in  $\tilde{T}_0$  are pairwise incompatible. Let  $\tau_n, n \in \omega$  be a one-to-one recursive enumeration of  $\tilde{T}_0$ . Put  $T = T_0 \cup \bigcup_n \{\tau_n \hat{\ } \sigma \mid \sigma \in T_n\}$ . Thus  $T$  is an infinite recursive subtree of  $2^{<\omega}$ . Let  $P = [T]$ , the set of paths through  $T$ . Thus  $P$  is a nonempty  $\Pi_1^0$  subset of  $2^\omega$ . By construction we have  $P = P_0 \cup \bigcup_n \{\tau_n \hat{\ } f \mid f \in P_n\}$ , hence  $P$  is Turing degree isomorphic to  $\bigcup_n P_n = S$ .  $\square$

**Remark 10.12.** In the proof of Theorem 10.11, note that  $\mathbf{p} = \inf_n \mathbf{p}_n$ , where  $\mathbf{p} = \text{deg}_w(P)$  and  $\mathbf{p}_n = \text{deg}_w(P_n)$ . Thus the proof shows that  $\mathcal{P}_w$  is closed under effective infima.

**Remark 10.13.** If  $C$  is a class of recursive functions satisfying the hypotheses of Theorem 10.10, put  $\mathbf{d}_C = \text{deg}_w(\text{DNR}_C)$ . We have seen that  $\mathbf{d}_C \in \mathcal{P}_w$  and that  $\mathbf{d}_C$  is natural in the sense that it can be defined in a way which does not depend on the choice of Gödel numbering. Moreover, if  $C^* \supset C$  is another such class, then  $\mathbf{d}_{C^*} \leq \mathbf{d}_C$ , and according to Ambos-Spies et al [2, Theorem 1.9] we have strict inequality  $\mathbf{d}_{C^*} < \mathbf{d}_C$  provided  $C^*$  contains a function which “grows much faster than” all functions in  $C$ . There are many examples and problems here.

**Example 10.14.** For each constructive ordinal  $\alpha$ , let  $C_\alpha$  be the class of recursive functions obtained at levels  $< \omega \cdot (1 + \alpha)$  of the transfinite Ackermann hierarchy. (See for instance Wainer [60].) Thus  $C_0$  is the class of primitive recursive functions,  $C_1$  is the class of functions which are primitive recursive

relative to the Ackermann function, etc. Putting  $\mathbf{d}_\alpha = \mathbf{d}_{C_\alpha}$  we have a transfinite descending sequence

$$\mathbf{d}_0 > \mathbf{d}_1 > \cdots > \mathbf{d}_\alpha > \mathbf{d}_{\alpha+1} > \cdots$$

in  $\mathcal{P}_w$ . Moreover, if  $\alpha$  is a limit ordinal, then  $\mathbf{d}_\alpha = \inf_{\beta < \alpha} \mathbf{d}_\beta$ . Thus we see a rich set of natural degrees in  $\mathcal{P}_w$  which are related to subrecursive hierarchies of the kind that arise in Gentzen-style proof theory.

**Remark 10.15.** Let us assume that we are using one of the standard Gödel numberings of Turing machines which appear in the literature. Then the function  $p(n)$  in the proof of Theorem 10.10 can be chosen to be bounded by a linear function. Therefore, instead of assuming that  $C$  is closed under composition with primitive recursive functions, we could assume merely that for all  $h \in C$  and  $c \geq 1$  there exists  $h_c^* \in C$  such that  $h_c^*(n) \geq h(m)$  for all  $m \leq c \cdot (n+1)$ . In particular, we can take  $C$  to be various well known computational complexity classes such as PTIME, EXPTIME, etc. For each such class, Theorem 10.10 shows that the weak degree  $\mathbf{d}_C \in \mathcal{P}_w$  is natural in that its definition does not depend on the choice of a standard Gödel numbering.

**Example 10.16.** In  $\mathcal{P}_w$  we have

$$\mathbf{d}_{\text{PTIME}} > \mathbf{d}_{\text{EXPTIME}} > \cdots$$

etc. Thus we see a rich set of natural degrees in  $\mathcal{P}_w$  which are related to computational complexity.

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