Mass problems and measure-theoretic regularity

Stephen G. Simpson
Department of Mathematics
Pennsylvania State University
http://www.math.psu.edu/simpson/

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Abstract

A well known fact is that every Lebesgue measurable set is *regular*, i.e., it includes an F_σ set of the same measure. We analyze this fact from a metamathematical or foundational standpoint. We study a family of Muchnik degrees corresponding to measure-theoretic regularity at all levels of the effective Borel hierarchy. We prove some new results concerning Nies's notion of LR-reducibility. We build some ω-models of RCA_0 which are relevant for the reverse mathematics of measure-theoretic regularity.

Contents

1 Introduction

Measure-theoretic regularity

Let S be a set in Euclidean space. Recall from classical analysis that S is said to be F_σ if and only if S is the union of a countable sequence of closed sets. In other words, $S = \bigcup_{i=0}^{\infty} C_i$ where each C_i is a closed set. Recall also that the *Borel sets* are the smallest family of sets which includes the closed sets and is closed under countable unions and complementation. It is well known that the Borel sets are arranged in a transfinite hierarchy according to how many times the countable union operation is iterated. It is well known that all Borel sets are *measurable* in the sense of Lebesgue.

A basic and well known theorem of measure theory reads as follows:

Theorem 1.1. *Every measurable set includes an F_σ set of the same measure.*

A variant theorem of measure theory is:

Theorem 1.2. *Every Borel set includes an F_σ set of the same measure.*

In this sense one sometimes says that Lebesgue measure is *regular*, or that Borel sets are *regular* with respect to Lebesgue measure. See for instance Halmos [15, Section 52]. This phenomenon is known as as *measure-theoretic regularity.*

Thr purpose of this paper is to calibrate the strength of Theorems 1.1 and 1.2 and their variants from a foundational standpoint. Roughly speaking, we quantify the "descriptive complexity" or "logical strength" of the F_σ sets which are needed in order to implement measure-theoretic regularity at various levels of the Borel hierarchy.

Our work in this paper contributes to two major streams of research in the foundations of mathematics: *degrees of unsolvability* and *reverse mathematics.* The purpose of this introductory section is to present the relevant background on degree theory, reverse mathematics, and measure-theoretic regularity.

Degrees of unsolvability

Degrees of unsolvability are a well known and highly developed research area which grew out of a fundamental discovery due to Turing 1936 [54]: the halting problem for Turing machines is algorithmically unsolvable. As is well known, Turing's example of an unsolvable mathematical problem was the first such example, and as such it revolutionized the foundations of mathematics. Subsequent research was motivated by the desire to discover additional examples of unsolvable mathematical problems and to quantify their unsolvability by classifying them according to the "amount" or *degree* of unsolvability which is inherent in them. See for instance Post 1944 [31] and Kleene/Post 1954 [20]. Later research during the period 1960–1990 was motivated by structural and methodological questions concerning various degree structures. Some classical treatises from this period are Sacks [33], Rogers [32], Shoenfield [36], Lerman [22], Soare [52], Odifreddi [29, 30].

The classical theory of degrees of unsolvability was concerned mainly with *decision problems* and their *Turing degrees.* A more recent trend has been to focus instead on *mass problems* and their *Muchnik degrees.* This modern direction in degrees of unsolvability has turned out to be especially fruitful for applications to various topics in the foundations of mathematics. Among these topics are reverse mathematics, intuitionism, algorithmic randomness, Kolmogorov complexity, resource bounded computational complexity, subrecursive hierarchies, and unsolvable mathematical problems. See in particular our recent papers [9, 41, 43, 44, 45, 46, 51] and our forthcoming treatise [49]. We emphasize that the study of mass problems offers a path along which the study of degrees of unsolvability can return to and reconnect with its roots in the foundations of mathematics.

In this paper we are concerned with two particular degree structures, $\mathcal{D}_{\mathrm{w}}$ and $\mathcal{E}_{\mathrm{w}}$, which are defined as follows.

Definition 1.3. For our purposes, a *Turing oracle* is a point in the *Baire space*, $\mathbb{N}^{\mathbb{N}}$, or the *Cantor space*, $2^{\mathbb{N}}$. A *mass problem* is a set of Turing oracles. Let P and Q be mass problems. We say that P *is weakly reducible to* Q, abbreviated $P \leq_{\mathrm{w}} Q$, if for each $Y \in Q$ there exists $X \in P$ such that X is computable using Y as a Turing oracle. The motivation behind this definition is that the set P is identified with the "problem" of finding at least one element of P. Thus P is "reducible" to Q if and only if each "solution" of Q can be used as a Turing oracle to compute a "solution" of P. We define a *weak degree* to be an equivalence class of mass problems under weak reducibility, i.e., under the equivalence relation $P \leq_{\mathrm{w}} Q$ and $Q \leq_{\mathrm{w}} P$. The weak degree of the mass problem P is denoted $\deg_{\mathrm{w}}(P)$. Weak degrees are also known as *Muchnik degrees.* The set of all weak degrees is partially ordered by letting $\deg_{\mathrm{w}}(P) \leq \deg_{\mathrm{w}}(Q)$ if and only if $P \leq_{\mathrm{w}} Q$. We define $\mathcal{D}_{\mathrm{w}}$ to be the partial ordering of all weak degrees. Writing $\mathbf{p} = \deg_{\mathrm{w}}(P)$ and $\mathbf{q} = \deg_{\mathrm{w}}(Q)$, the lattice operations in $\mathcal{D}_{\mathrm{w}}$ are given by $\inf(\mathbf{p}, \mathbf{q}) = \deg_{\mathrm{w}}(P \cup Q)$ and $\sup(\mathbf{p}, \mathbf{q}) = \deg_{\mathrm{w}}(P \times Q)$.

Definition 1.4. A mass problem P is said to be *effectively closed* if it is Π^0_1, i.e., P is the complement of the union of a computable list of basic open sets in $\mathbb{N}^{\mathbb{N}}$. We define $\mathcal{E}_{\mathrm{w}}$ to be the set of weak degrees associated with nonempty, effectively closed sets in the Cantor space, $2^{\mathbb{N}}$. Thus $\mathcal{E}_{\mathrm{w}}$ is a sublattice of $\mathcal{D}_{\mathrm{w}}$. Note that our restriction to the Cantor space is essential. We use $\mathbf{1}$ and $\mathbf{0}$ to denote the top and bottom degrees in $\mathcal{E}_{\mathrm{w}}$.

Remark 1.5. Historically, the study of mass problems and $\mathcal{D}_{\mathrm{w}}$ originated in considerations of Kolmogorov [21], Medvedev [24], and Muchnik [25] concerning the so-called "intuitionistic calculus of problems" [21]. In particular Muchnik [25] showed that $\mathcal{D}_{\mathrm{w}}$ is a complete Brouwerian lattice. Some recent papers on $\mathcal{E}_{\mathrm{w}}$ are [2, 4, 9, 41, 43, 44, 45, 46]. In these papers $\mathcal{E}_{\mathrm{w}}$ is often denoted $\mathcal{P}_{\mathrm{w}}$. Lemma 6.4 below implies that $\mathcal{E}_{\mathrm{w}}$ includes a large and significant part of $\mathcal{D}_{\mathrm{w}}$. For additional historical references see [45].

In this paper we study certain mass problems associated with measure-theoretic regularity. Several of our results may be summarized as follows. To each level of the effective Borel hierarchy we associate a specific, natural degree of unsolvability in $\mathcal{D}_{\rm w}$. Namely, for each recursive ordinal number α let $\mathbf{b}_\alpha$ be the Muchnik degree associated with the problem of "regularizing" sets at level $\alpha+2$ of the effective Borel hierarchy. Thus we have

$$\mathbf{b}_\alpha = \deg_{\rm w}(\{Y \mid \text{every } \Sigma^0_{\alpha+2} \text{ set includes a } \Sigma^{0,Y}_2 \text{ set of the same measure}\}).$$

It turns out that the Muchnik degrees $\inf(\mathbf{b}_\alpha, \mathbf{1})$ belong to $\mathcal{E}_{\rm w}$ and are distinct from one another. In this way we obtain a mathematically natural embedding of the hyperarithmetical hierarchy into $\mathcal{E}_{\rm w}$. This embedding is different from, and foundationally more relevant than, the one in [9, Section 4]. The details of our new embedding are in Section 6 below.

Reverse mathematics

Reverse mathematics is a well known, highly developed research program in the foundations of mathematics. The purpose of reverse mathematics is to classify specific mathematical theorems up to logical equivalence according to the strength of the set existence axioms which are needed to prove them. These axioms are embodied in certain formal, deductive systems. The most important formal systems for reverse mathematics are the so-called "Big Five": RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0, corresponding to Chapters II–VI of [40, 47]. The standard reference for reverse mathematics is Simpson [40, 47]. See also our recent survey in [50].

The present paper includes a contribution to the reverse mathematics of measure theory. In order to place this contribution in context, we now briefly outline the previous research in this area.

The first wave of research in the reverse mathematics of measure theory dealt with additivity properties and was centered around the system WWKL_0. This was initiated in the 1980s by Yu [55] and continued in Yu/Simpson [60], Yu [56, 57, 58, 59], and Brown/Giusto/Simpson [5]. Recall the principal axiom of WKL_0, which says that any tree T containing bitstrings of length n for each $n \in \mathbb{N}$ has an infinite path. The principal axiom of WWKL_0 is weaker. It says that T has an infinite path provided $\exists \epsilon\, \forall n\, (|T \cap 2^n|/2^n > \epsilon > 0)$, i.e., the fraction of bitstrings of length n belonging to T is bounded away from 0. It was shown in the 1980s and 1990s that WWKL_0 is necessary and sufficient in order to prove many basic principles of measure theory, including a version of countable additivity and a version of the Vitali Covering Lemma. Subsequently it was noticed that WWKL_0 is closely related to algorithmic randomness in the sense of Martin-Löf [23, 28, 11, 48]. Indeed, the principal axiom of WWKL_0 turned out to be equivalent over RCA_0 to the statement

$$\forall X\, \exists Y\, (Y \text{ is Martin-Löf random relative to } X).$$

See also our summary in [40, 47, Section X.1]. Later Simpson [38, 39, 41] developed the relationship to mass problems. For instance, the Muchnik degree

$$\mathbf{r} = \deg_{\rm w}(\{X \mid X \text{ is Martin-Löf random}\})$$

was the first example of a specific, natural degree in $\mathcal{E}_{\mathrm{w}}$ other than $\mathbf{0}$ and $\mathbf{1}$.

The second wave dealing with measure-theoretic regularity was initiated in 2002 by Dobrinen/Simpson [10] and was centered around our notion of almost everywhere domination. In [10] we defined a Turing oracle Y to be *almost everywhere dominating* if for all Turing oracles X except a set of measure 0, every function $f : \mathbb{N} \to \mathbb{N}$ which is computable using X is dominated by a function $g : \mathbb{N} \to \mathbb{N}$ which is computable using Y. In [10] and [18] it emerged that Y is almost everywhere dominating if and only if Y suffices to "regularize" every set at level 3 of the effective Borel hierarchy. In addition, a close connection with Nies's notion of LR-reducibility was discovered. Thus Y is almost everywhere dominating if and only if every Σ^0_3 set includes a $\Sigma^{0,Y}_2$ set of the same measure, if and only if $0' \leq_{\mathrm{LR}} Y$. Here $0'$ denotes the halting problem for Turing machines. See Dobrinen/Simpson [10], Binns/Kjos-Hanssen/Lerman/Solomon [3], Cholak/Greenberg/Miller [7], Kjos-Hanssen [17], Kjos-Hanssen/Miller/Solomon [18], and our exposition in [42]. The relationship between almost everywhere domination and mass problems was developed in Kjos-Hanssen [17] and Simpson [44]. In particular, it was shown that the Muchnik degree $\inf(\mathbf{b}, \mathbf{1})$ where

$$\mathbf{b} = \deg_{\mathrm{w}}(\{Y \mid Y \text{ is almost everywhere dominating}\})$$

belongs to $\mathcal{E}_{\mathrm{w}}$ and is incomparable with $\mathbf{r}$.

In this paper we continue and expand the second wave. Namely, we generalize the theory of almost everywhere domination from $0'$ to the entire hyperarithmetical hierarchy, with corresponding applications to the metamathematics of measure-theoretic regularity. The details of this generalization are in Sections 3 and 4 and 6 below. In particular, our Theorems 4.11 and 6.6 for arbitrary recursive ordinals α were first proved in [10] and [18] for the special case $\alpha = 1$.

In addition, we use our results concerning degrees of unsolvability to build models of RCA_0 which are relevant for the reverse mathematics of measure-theoretic regularity. We obtain models M_1, M_2, M_3, M_4 satisfying $\mathsf{RCA}_0 + \neg\,\mathsf{WWKL}_0$ and $\mathsf{WWKL}_0 + \neg\,\mathsf{WKL}_0$ and $\mathsf{WKL}_0 + \neg\,\mathsf{ACA}_0$ and $\mathsf{ACA}_0 + \neg\,\mathsf{ATR}_0$ respectively such that each of these models satisfies a kind of measure-theoretic regularity at all levels of the Borel hierarchy. The details are in Section 7 below.

2 Notation and preliminaries

In this section we briefly review some well known concepts and notation from recursion theory.

Definition 2.1 (Baire space, Cantor space)**.** We use standard recursion-theoretic notation from Rogers [32]. We use letters such as $i, j, k, l, m, n, \ldots$ to denote natural numbers. We use $\mathbb{N}$ to denote the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$. We use letters such as $I, J, \ldots$ to denote subsets of $\mathbb{N}$. We use $\mathbb{R}$ to denote the set of real numbers. We use $\mathbb{N}^{\mathbb{N}}$ to denote the *Baire space*, $\mathbb{N}^{\mathbb{N}} = \{X \mid X : \mathbb{N} \to \mathbb{N}\}$. We use $2^{\mathbb{N}}$ to denote the *Cantor space*, $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}} = \{X \mid X : \mathbb{N} \to \{0,1\}\}$. We use letters such as $X, Y, Z, A, B, C, \ldots$

to denote *Turing oracles*, i.e., points of the Baire space. We use letters such as $P, Q, R, S, \ldots$ to denote *mass problems*, i.e., subsets of the Baire space. Note also that the Cantor space is a subspace of the Baire space.

Definition 2.2 (Π^0_1 predicates, Σ^0_3 predicates)**.** A predicate $R \subseteq (\mathbb{N}^\mathbb{N})^k \times \mathbb{N}^l$ is said to be *recursive* if and only if its characteristic function $\chi_R : (\mathbb{N}^\mathbb{N})^k \times \mathbb{N}^l \to \{0,1\}$ is Turing computable. For $X \in \mathbb{N}^\mathbb{N}$ we say that R is *X-recursive* or *recursive relative to* X if χ_R is Turing computable using the oracle X. A predicate $P \subseteq (\mathbb{N}^\mathbb{N})^k \times \mathbb{N}^l$ is said to be Π^0_1 if and only if there exists a recursive predicate $R \subseteq (\mathbb{N}^\mathbb{N})^k \times \mathbb{N}^{l+1}$ such that $P(-,-) \equiv \forall n\, R(-,-,n)$. We say that P is $\Pi^{0,X}_1$ if there exists an X-recursive predicate R such that $P(-,-) \equiv \forall n\, R(-,-,n)$. A predicate $S \subseteq (\mathbb{N}^\mathbb{N})^k \times \mathbb{N}^l$ is said to be Σ^0_3 if and only if there exists a recursive predicate $R \subseteq (\mathbb{N}^\mathbb{N})^k \times \mathbb{N}^{l+3}$ such that $S(-,-) \equiv \exists k\, \forall m\, \exists n\, R(-,-,k,m,n)$. Other levels of the arithmetical hierarchy are defined similarly. See [32, Chapters 14 and 15] and Definition 3.1 below. A set $I \subseteq \mathbb{N}$ is *X-recursively enumerable*, abbreviated *X-r.e.*, if and only if I is $\Sigma^{0,X}_1$, i.e., $I = \{i \mid \exists n\, R(i,n)\}$ for some X-recursive predicate $R \subseteq \mathbb{N}^2$.

Definition 2.3 (strings and bitstrings)**.** A *string* is a finite sequence of natural numbers. A *bitstring* is a string of 0's and 1's. We use letters such as $\sigma, \tau, \ldots$ to denote strings and bitstrings. Let $|\sigma|$ be the length of σ. The domain of σ is a finite initial segment of $\mathbb{N}$, denoted $\mathrm{dom}(\sigma) = \{n \mid n < |\sigma|\}$. We have $\sigma = \langle i_0, i_1, \ldots, i_{|\sigma|-1}\rangle$ where $i_n = \sigma(n)$. The range of σ is a finite subset of $\mathbb{N}$, denoted $\mathrm{rng}(\sigma) = \{\sigma(n) \mid n < |\sigma|\}$. Let $\sigma^\frown\tau$ be the *concatenation*, σ followed by τ. Thus $|\sigma^\frown\tau| = |\sigma| + |\tau|$ and $\mathrm{rng}(\sigma^\frown\tau) = \mathrm{rng}(\sigma) \cup \mathrm{rng}(\tau)$. For $X \in \mathbb{N}^\mathbb{N}$ and $s \in \mathbb{N}$ let $X \upharpoonright s = \langle X(0), X(1), \ldots, X(s-1)\rangle$, a string of length s. We write $\sigma \subset X$ if and only if $\sigma = X \upharpoonright |\sigma|$. We write $\sigma \subseteq \tau$ if and only if $\sigma = \tau$ or σ is a proper initial segment of τ.

Definition 2.4 (oracle computations)**.** For $e, n, i \in \mathbb{N}$ we write $\varphi^{(1)}_e(n)\downarrow= i$ if and only if the Turing machine with Gödel number e and input n eventually halts with output i. For $X \in \mathbb{N}^\mathbb{N}$ we write $\varphi^{(1),X}_e(n)\downarrow= i$ if and only if the Turing machine with Gödel number e and input n using X as an oracle eventually halts with output i. We write $\varphi^{(1),X}_e(n)\downarrow$ if and only if $\varphi^{(1),X}_e(n)\downarrow= i$ for some i, otherwise $\varphi^{(1),X}_e(n)\uparrow$. For $s \in \mathbb{N}$ we write $\varphi^{(1),X\upharpoonright s}_{e,s}(n)\downarrow= i$ if and only if the Turing machine with Gödel number e and input n using X as an oracle halts in $< s$ steps with output i using only oracle information from $X \upharpoonright s$. For a string σ we write $\varphi^{(1),\sigma}_{e,s}(n)\downarrow= i$ if and only if $\varphi^{(1),X\upharpoonright s}_{e,s}(n)\downarrow= i$ where $\sigma = X \upharpoonright s$. Note that the 5-place predicate $\varphi^{(1),\sigma}_{e,s}(n)\downarrow= i$ is recursive, and $\varphi^{(1),X}_e(n)\downarrow= i$ if and only if $\varphi^{(1),X\upharpoonright s}_{e,s}(n)\downarrow= i$ for some s. We write $W^X_e = \{n \mid \varphi^{(1),X}_e(n)\downarrow\}$ and $W^X_{e,s} = \{n \mid \varphi^{(1),X\upharpoonright s}_{e,s}(n)\downarrow\}$. Note that $W^X_e = \bigcup_{s=0}^\infty W^X_{e,s}$, and W^X_e for $e = 0, 1, 2, \ldots$ is a uniform recursive enumeration of all of the X-r.e. sets.

Definition 2.5 (Turing reducibility, Muchnik reducibility)**.** We say that X is *recursive* if X is Turing computable, i.e., $\exists e\, \forall n\, (\varphi^{(1)}_e(n)\downarrow= X(n))$. We say that X *is Turing reducible to* Y, abbreviated $X \leq_\mathrm{T} Y$, if X is Y-recursive, i.e., $\exists e\,\forall n\,(\varphi_e^{(1),Y}(n)\downarrow = X(n))$. We say that X and Y are *Turing equivalent* if $X \le_{\mathrm{T}} Y$ and $Y \le_{\mathrm{T}} X$. We sometimes use 0 to denote the constant function $0 \in \mathbb{N}^{\mathbb{N}}$, i.e., $0(n) = 0$ for all $n \in \mathbb{N}$. Thus X is recursive if and only if $X \le_{\mathrm{T}} 0$. The *pairing function* $\oplus : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is defined by $(X \oplus Y)(2n) = X(n)$ and $(X \oplus Y)(2n+1) = Y(n)$. Thus $X \oplus Y \le_{\mathrm{T}} Z$ if and only if $X \le_{\mathrm{T}} Z$ and $Y \le_{\mathrm{T}} Z$. For $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$ we say that *P is weakly reducible to Q*, abbreviated $P \le_{\mathrm{w}} Q$, if for all $Y \in Q$ there exists $X \in P$ such that $X \le_{\mathrm{T}} Y$. Weak reducibility is also known as *Muchnik reducibility.* See also Definitions 1.3 and 1.4 above. We write $P \times Q = \{X \oplus Y \mid X \in P \text{ and } Y \in Q\}$. Thus $P \times Q \le_{\mathrm{w}} S$ if and only if $P \le_{\mathrm{w}} S$ and $Q \le_{\mathrm{w}} S$.

Definition 2.6 (the Turing jump operator)**.** We define an operator $X \mapsto X' : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$X'(n) \;=\; \begin{cases} \varphi_n^{(1),X}(n) + 1 & \text{if } \varphi_n^{(1),X}(n)\downarrow, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.7. Note that our version of the Turing jump operator in Definition 2.6 is somewhat unusual. Of course, our X' is uniformly Turing equivalent to the usual Turing jump of X as defined for instance in Rogers [32]. An advantage of our $X \mapsto X'$ over the usual Turing jump operator is expressed in the following lemma. See also Simpson [42, Remark 8.7] and Cole/Simpson [9, Lemma 2.5].

Lemma 2.8. *If $X \in \mathbb{N}^{\mathbb{N}}$ is a Π^0_1 singleton, then so is X'. More generally, if $P \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^0_1 then so is $P' = \{X' \mid X \in P\}$.*

Proof. If $\varphi_n^{(1),X}(n)\downarrow$ let $\psi^X(n) =$ the least s such that $\varphi_{n,s}^{(1),X\restriction s}(n)\downarrow$. By the S-m-n Theorem let $p(n)$ be a recursive function such that $X'(p(n)) = \psi^X(n)+1$ if $\varphi_n^{(1),X}(n)\downarrow$, and $X'(p(n)) = 0$ otherwise, for all X and all n. Let $q(n)$ be a recursive function such that $X'(q(n)) = X(n)+1$, for all X and all n. Since P is Π^0_1, fix an e such that $P = \{X \mid \varphi_e^{(1),X}(e)\uparrow\}$. Then $Z \in P'$ if and only if (a) $\varphi_{e,s}^{(1),\langle Z(q(0))-1, Z(q(1))-1,\ldots,Z(q(s-1))-1\rangle}(e)\uparrow$ for all s, and (b) for all n either $Z(n) = 0$ and $\varphi_{n,s}^{(1),\langle Z(q(0))-1, Z(q(1))-1,\ldots,Z(q(s-1))-1\rangle}(n)\uparrow$ for all s, or $Z(n) > 0$ and $Z(p(n)) > 0$ and $Z(p(n)) - 1 =$ the least s such that $\varphi_{n,s}^{(1),\langle Z(q(0))-1, Z(q(1))-1,\ldots,Z(q(s-1))-1\rangle}(n)\downarrow = Z(n)-1$. Thus P' is Π^0_1. □

Definition 2.9 (recursive ordinals, hyperarithmetical hierarchy)**.** A *recursive ordinal* is an ordinal number which is the order type of a recursive well ordering of a subset of $\mathbb{N}$. The least nonrecursive ordinal is denoted ω_1^{CK}. More generally, an ordinal is said to be *X-recursive* if it is the order type of an X-recursive well ordering of a subset of $\mathbb{N}$. The least ordinal which is not X-recursive is denoted ω_1^X. We use letters such as $\alpha, \beta, \ldots$ to denote ordinals. Given an X-recursive ordinal α, it is possible to iterate the Turing jump operator α times starting with X. The resulting oracle $X^{(\alpha)}$ is well defined up to Turing equivalence. Thus we have $X^{(0)} = X$, $X^{(1)} = X'$, $X^{(2)} = X''$, …, $X^{(\alpha)}$, $X^{(\alpha+1)}$, … . This transfinite sequence of oracles is known as the *hyperarithmetical hierarchy* relative to X. See for instance Sacks [34, Part A] and Simpson [40, 47, Section VIII.3].

Definition 2.10 (Π^0_1 singletons, Σ^0_3 singletons)**.** A Π^0_1 *singleton* is any $X \in \mathbb{N}^\mathbb{N}$ such that the one-element set $\{X\}$ is Π^0_1. More generally, given a Turing oracle A, a $\Pi^{0,A}_1$ *singleton* is any $X \in \mathbb{N}^\mathbb{N}$ such that $\{X\}$ is $\Pi^{0,A}_1$. Similarly we can define what we mean by a Σ^0_3 *singleton* and a $\Sigma^{0,A}_3$ *singleton*. It is well known that $X^{(\alpha)}$ for each $\alpha < \omega^X_1$ is a $\Sigma^{0,X}_3$ singleton. Also, every Σ^0_3 singleton is Turing equivalent to a Π^0_1 singleton. See for instance the first paragraph of the proof of Lemma 4.6 below.

Definition 2.11 (diagonal nonrecursiveness, PA-completeness)**.** We say that $X \in \mathbb{N}^\mathbb{N}$ is *diagonally nonrecursive*, abbreviated DNR, if there is no n such that $\varphi^{(1)}_n(n) \downarrow= X(n)$. We sometimes write

$$\text{DNR} = \{X \in \mathbb{N}^\mathbb{N} \mid X \text{ is diagonally nonrecursive}\}.$$

Let PA be the set of all complete, consistent extensions of first-order Peano Arithmetic. It is well known that PA is Muchnik equivalent to $\text{DNR} \cap 2^\mathbb{N}$. Given $X \in \mathbb{N}^\mathbb{N}$ let PA^X be the set of all complete, consistent extensions of first-order Peano arithmetic with an additional 1-place function symbol $\underline{X}$ and additional axioms $\underline{X}(\underline{n}) = \underline{m}$ for all $n, m \in \mathbb{N}$ such that $X(n) = m$. We say that Y *is* PA-*complete over* X if $\text{PA}^X \leq_\text{w} \{Y\}$. By the Kleene Basis Theorem [19, Theorem 38*, pages 401–402] we know that X' is PA-complete over X.

Definition 2.12 (Martin-Löf randomness)**.** The *fair coin measure* is the countably additive Borel measure μ on $2^\mathbb{N}$ defined by $\mu(N_\sigma) = 1/2^{|\sigma|}$ for all bitstrings σ. Here $N_\sigma = \{X \in 2^\mathbb{N} \mid \sigma \subset X\}$. Letting $U_e = \{Z \in 2^\mathbb{N} \mid \varphi^{(1),Z}_e(e) \downarrow\}$, we say that $Z \in 2^\mathbb{N}$ is *Martin-Löf random* if $Z \notin \bigcap^\infty_{n=0} U_{p(n)}$ whenever $p(n)$ is a recursive function such that $\mu(U_{p(n)}) \leq 1/2^n$ for all n. More generally, letting $U^X_e = \{Z \in 2^\mathbb{N} \mid \varphi^{(1),X\oplus Z}_e(e) \downarrow\}$, we say that Z *is Martin-Löf random relative to* X if $Z \notin \bigcap^\infty_{n=0} U^X_{p(n)}$ whenever $p(n)$ is a recursive function such that $\mu(U^X_{p(n)}) \leq 1/2^n$ for all n. For much more information on Martin-Löf's concept of randomness, see Nies [28] and Downey/Hirschfeldt [11] and Simpson [48].

3 Measure-theoretic regularity

In this section we review and generalize a well known result concerning measure-theoretic regularity in the effective Borel hierarchy. We also review some related recent results concerning LR-reducibility.

Definition 3.1 (The effective Borel hierarchy)**.** Given a Turing oracle X and an X-recursive ordinal $\alpha < \omega^X_1$, we define what it means for a set $S \subseteq 2^\mathbb{N}$ to be $\Sigma^{0,X}_\alpha$ or $\Pi^{0,X}_\alpha$. For $\alpha = 0$ we define $\Sigma^{0,X}_0 = \Pi^{0,X}_0 =$ the class of clopen sets in $2^\mathbb{N}$. Recall that $S \subseteq 2^\mathbb{N}$ is clopen if and only if it is the union of a finite sequence of basic open sets. Such sets are recursively indexed in an obvious way. For successor ordinals, we define S to be $\Sigma^{0,X}_{\alpha+1}$ if and only if $S = \bigcup^\infty_{i=0} P_i$ where each P_i is $\Pi^{0,X}_\alpha$ via an index which is X-recursive as a function of i. In this case an *index* of S consists of an X-recursive index of such a function plus an X-recursive notation for α. We define P to be $\Pi^{0,X}_\alpha$ if and only if $2^\mathbb{N} \setminus P$ is $\Sigma^{0,X}_\alpha$. For limit ordinals α we define $\Sigma^{0,X}_\alpha = \bigcup_{\beta<\alpha} \Sigma^{0,X}_\beta$.

Remark 3.2 (lightface versus boldface)**.** (1) Let α be a recursive ordinal. A set $S \subseteq 2^\mathbb{N}$ is said to be Σ^0_α or *lightface* Σ^0_α if and only if S is $\Sigma^{0,0}_\alpha$. This special case $X = 0$ is known as the *lightface Borel hierarchy*. By diagonalization one can construct a $\Sigma^0_{\alpha+1}$ set which is neither $\Sigma^{0,X}_\alpha$ nor $\Pi^{0,X}_\alpha$ for any X. (2) Let α be a countable ordinal. A set $S \subseteq 2^\mathbb{N}$ is said to be *boldface* Σ^0_α or *at level α of the Borel hierarchy* if and only if S is $\Sigma^{0,X}_\alpha$ for some X. It is well known that each Borel set is boldface Σ^0_α for some countable ordinal α. It is well known that S is F_σ if and only if S is boldface Σ^0_2.

We now discuss measure-theoretic regularity in the effective Borel hierarchy. We refer to the fair coin measure, Definition 2.12.

Theorem 3.3. *Let α be an X-recursive ordinal. Every $\Sigma^{0,X}_{\alpha+2}$ subset of $2^\mathbb{N}$ includes a $\Sigma^{0,X^{(\alpha)}}_2$ set of the same measure. Conversely, every $\Sigma^{0,X^{(\alpha)}}_2$ set is $\Sigma^{0,X}_{\alpha+2}$.*

Proof. For finite ordinals $\alpha = n < \omega$, this result is due to Kautz [16, Lemma II.1.3]. The generalization to arbitrary X-recursive ordinals α is routine. $\square$

The next definition is due to Nies [26, Section 8].

Definition 3.4 (LR-reducibility)**.** Let X and Y be Turing oracles. We write $X \leq_{\mathrm{LR}} Y$ if and only if every $Z \in 2^\mathbb{N}$ which is Martin-Löf random relative to Y is Martin-Löf random relative to X. Note that $X \leq_{\mathrm{T}} Y$ implies $X \leq_{\mathrm{LR}} Y$, but the converse does not hold.

Theorem 3.5. *The following are pairwise equivalent.*

1. *$X \leq_{\mathrm{LR}} Y$.*
2. *Every $\Pi^{0,X}_1$ subset of $2^\mathbb{N}$ of positive measure includes a $\Pi^{0,Y}_1$ set of positive measure.*
3. *Given $\epsilon > 0$ and a $\Pi^{0,X}_1$ set $P \subseteq 2^\mathbb{N}$, we can find a $\Pi^{0,Y}_1$ set $Q \subseteq P$ such that $\mu(P \setminus Q) < \epsilon$.*

Proof. This result is due to Kjos-Hanssen [17]. See also our exposition in [42, Theorem 4.6]. $\square$

Theorem 3.6. *The following are pairwise equivalent.*

1. *$X \leq_{\mathrm{LR}} Y$ and $X \leq_{\mathrm{T}} Y'$.*
2. *Every $\Pi^{0,X}_1$ subset of $2^\mathbb{N}$ includes a $\Sigma^{0,Y}_2$ set of the same measure.*
3. *Every $\Sigma^{0,X}_2$ subset of $2^\mathbb{N}$ includes a $\Sigma^{0,Y}_2$ set of the same measure.*

Proof. This result is due to Kjos-Hanssen/Miller/Solomon [18]. See also our exposition in [42, Theorem 5.13, Remark 7.1]. □

Remark 3.7. Theorems 3.3 and 3.5 and 3.6 imply a close relationship between measure-theoretic regularity and LR-reducibility. A sharper version of this relationship will be stated in Theorem 4.11 below. In order to prove Theorem 4.11 we shall need another result on LR-reducibility, namely, Theorem 4.9 below.

4 LR-reducibility

In this section we prove a new result concerning LR-reducibility. Namely,

$$X^{(\alpha)} \leq_{\mathrm{LR}} Y \text{ implies } X^{(\alpha+1)} \leq_{\mathrm{T}} X \oplus Y'.$$

We then use this result to sharpen the relationship between LR-reducibility and measure-theoretic regularity.

There are several equivalent characterizations of LR-reducibility. We shall rely on the characterization in Theorem 4.3 below.

Definition 4.1 (computable measures)**.** Let $\mu : \mathbb{N} \to \mathbb{R}^+$ be any computable function from the natural numbers to the positive real numbers. We extend μ to a *computable measure* on $\mathbb{N}$ by defining $\mu(I) = \sum_{i \in I} \mu(i)$ for all $I \subseteq \mathbb{N}$.

The next definition is due to Nies [26, Section 8].

Definition 4.2 (LK-reducibility)**.** We write $X \leq_{\mathrm{LK}} Y$ if and only if $\mathrm{K}^Y(i) \leq \mathrm{K}^X(i) + O(1)$ for all i. In other words, there exists a constant c such that $\mathrm{K}^Y(i) \leq \mathrm{K}^X(i)+c$ for all i. Here K^X denotes prefix-free Kolmogorov complexity relative to the Turing oracle X.

Theorem 4.3. *The following are pairwise equivalent.*

1. $X \leq_{\mathrm{LR}} Y$.
2. $X \leq_{\mathrm{LK}} Y$.
3. *For each computable measure μ and X-r.e. set I such that $\mu(I) < \infty$, we can find a Y-r.e. set J such that $\mu(J) < \infty$ and $I \subseteq J$.*

Proof. This result is due to Kjos-Hanssen/Miller/Solomon [18]. See also our exposition in [42]. □

The next definition and lemma are due to Nies [27] and Simpson [42, Definition 8.3, Lemma 8.4].

Definition 4.4 (jump-traceability)**.** Recall Definition 2.6 where we defined the Turing jump operator $X \mapsto X'$ in a somewhat unusual manner. We say that X *is weakly jump-traceable by* Y if there exists a total recursive function $p(n)$ such that $\forall n\,(X'(n) \in W^Y_{p(n)}$ and $W^Y_{p(n)}$ is finite). We say that X *is jump-traceable by* Y if there exist total recursive functions $p(n)$ and $q(n)$ such that $\forall n\,(X'(n) \in W^Y_{p(n)}$ and $|W^Y_{p(n)}| \leq q(n))$. Here $q(n)$ is called a *bounding function.*

Lemma 4.5. *If $X \leq_{\mathrm{LR}} Y$ then X is jump-traceable by Y with bounding function $q(n) = 2^{c+n}$ for some constant c.*

Proof. Consider the X-r.e. set $I = \{(n,i) \mid X'(n) = i \text{ or } i = 0\}$. Clearly $\sum_{(n,i)\in I} 1/2^n \leq 4 < \infty$. By Theorem 4.3 let J be a Y-r.e. set such that $\sum_{(n,j)\in J} 1/2^n < \infty$ and $I \subseteq J$. Let $p(n)$ be a recursive function such that $W^Y_{p(n)} = \{j \mid (n,j) \in J\}$. Let c be such that $\sum_{(n,j)\in J} 1/2^n \leq 2^c$. It follows easily that $|W^Y_{p(n)}| \leq 2^{c+n}$, Q.E.D. □

We view the next lemma and theorem as vast generalizations of [42, Lemma 8.5, Theorem 8.8].

Lemma 4.6. *Assume that $A \leq_{\mathrm{T}} X$ and that X is a $\Sigma^{0,A}_3$ singleton. If X is weakly jump-traceable by Y, then $X' \leq_{\mathrm{T}} A \oplus Y'$.*

Proof. Since X is a $\Sigma^{0,A}_3$ singleton, let $R(Z,k,m,n)$ be an A-recursive predicate such that X is the unique Z satisfying $\exists k\, \forall m\, \exists n\, R(Z,k,m,n)$. Fix a k satisfying $\forall m\, \exists n\, R(X,k,m,n)$. Define $f : \mathbb{N} \to \mathbb{N}$ by letting $f(m) =$ the least n satisfying $R(X,k,m,n)$. Then $X \oplus f$ is a $\Pi^{0,A}_1$ singleton, being the unique $Z \oplus g$ satisfying $\forall m\, (g(m) = \text{the least } n \text{ satisfying } R(Z,k,m,n))$. Moreover $X \oplus f \leq_{\mathrm{T}} X \oplus A \leq_{\mathrm{T}} X$, hence $X \oplus f$ is weakly jump-traceable by Y. Thus, replacing X by $X \oplus f$, we may safely assume that X is a $\Pi^{0,A}_1$ singleton.

Since X is a $\Pi^{0,A}_1$ singleton, it follows by Lemma 2.8 relativized to A that X' is a $\Pi^{0,A}_1$ singleton. Let $Q(Z)$ be a $\Pi^{0,A}_1$ predicate such that X' is the unique Z satisfying $Q(Z)$. Since X is weakly jump-traceable by Y, let $F_n = W^Y_{p(n)}$ where $p(n)$ is as in Definition 4.4. Then F_n for $n = 0,1,2,\ldots$ is a Y'-recursive sequence of finite sets, and $X'(n) \in F_n$ for all n. Thus

$$\{Z \mid Q(Z) \text{ and } \forall n\, (Z(n) \in F_n)\}$$

is a Y'-recursively bounded $\Pi^{0,A}_1$ set whose only member is X'. It follows that $X' \leq_{\mathrm{T}} A \oplus Y'$, Q.E.D. □

Theorem 4.7. *Assume that $A \leq_{\mathrm{T}} X$ and that X is a $\Sigma^{0,A}_3$ singleton. Then $X \leq_{\mathrm{LR}} Y$ implies $X' \leq_{\mathrm{T}} A \oplus Y'$.*

Proof. This is immediate from Lemmas 4.5 and 4.6. □

Corollary 4.8. *Let X be a Σ^0_3 singleton. Then $X \leq_{\mathrm{LR}} Y$ implies $X' \leq_{\mathrm{T}} Y'$.*

Proof. This is the special case $A = 0$ of Theorem 4.7. □

Theorem 4.9. *For each $\alpha < \omega^X_1$, $X^{(\alpha)} \leq_{\mathrm{LR}} Y$ implies $X^{(\alpha+1)} \leq_{\mathrm{T}} X \oplus Y'$.*

Proof. This follows from Theorem 4.7 plus the well known fact that $X^{(\alpha)}$ is a $\Sigma^{0,X}_3$ singleton. For a proof of this fact, see any textbook of hyperarithmetical theory, e.g., Ash/Knight [1], Rogers [32, Chapter 16], Sacks [34, Part A], Shoenfield [35, Sections 7.8–7.11], Simpson [40, 47, Section VIII.3]. □

Corollary 4.10. *For each $\alpha < \omega_1^{\mathrm{CK}}$, $0^{(\alpha)} \leq_{\mathrm{LR}} Y$ implies $0^{(\alpha+1)} \leq_{\mathrm{T}} Y'$.*

Proof. This is the special case $X = 0$ of Theorem 4.9. $\square$

Our sharp theorem concerning measure-theoretic regularity reads as follows.

Theorem 4.11. *For each $\alpha < \omega_1^X$ the following are pairwise equivalent.*

1. *$X^{(\alpha)} \leq_{\mathrm{LR}} Y$ and $X \leq_{\mathrm{T}} Y'$.*
2. *Every $\Pi_1^{0,X^{(\alpha)}}$ subset of $2^{\mathbb{N}}$ includes a $\Sigma_2^{0,Y}$ set of the same measure.*
3. *Every $\Sigma_2^{0,X^{(\alpha)}}$ subset of $2^{\mathbb{N}}$ includes a $\Sigma_2^{0,Y}$ set of the same measure.*
4. *Every $\Pi_{\alpha+1}^{0,X}$ subset of $2^{\mathbb{N}}$ includes a $\Sigma_2^{0,Y}$ set of the same measure.*
5. *Every $\Sigma_{\alpha+2}^{0,X}$ subset of $2^{\mathbb{N}}$ includes a $\Sigma_2^{0,Y}$ set of the same measure.*

Proof. By Theorems 3.3 and 3.6, each of conditions 2 through 5 is equivalent to the conjunction of $X^{(\alpha)} \leq_{\mathrm{LR}} Y$ and $X^{(\alpha)} \leq_{\mathrm{T}} Y'$. But then by Theorem 4.9 we can weaken $X^{(\alpha)} \leq_{\mathrm{T}} Y'$ to $X \leq_{\mathrm{T}} Y'$. Thus condition 5 is equivalent to condition 1, Q.E.D. $\square$

Corollary 4.12. *For each $\alpha < \omega_1^{\mathrm{CK}}$ the following are pairwise equivalent.*

1. *$0^{(\alpha)} \leq_{\mathrm{LR}} Y$.*
2. *Every $\Pi_1^{0,0^{(\alpha)}}$ subset of $2^{\mathbb{N}}$ includes a $\Sigma_2^{0,Y}$ set of the same measure.*
3. *Every $\Sigma_2^{0,0^{(\alpha)}}$ subset of $2^{\mathbb{N}}$ includes a $\Sigma_2^{0,Y}$ set of the same measure.*
4. *Every $\Pi_{\alpha+1}^{0}$ subset of $2^{\mathbb{N}}$ includes a $\Sigma_2^{0,Y}$ set of the same measure.*
5. *Every $\Sigma_{\alpha+2}^{0}$ subset of $2^{\mathbb{N}}$ includes a $\Sigma_2^{0,Y}$ set of the same measure.*

Proof. This is the special case $X = 0$ of Theorem 4.11. $\square$

5 DNR avoidance and cone avoidance

We say that Y *avoids* DNR if DNR $\not\leq_{\mathrm{w}} \{Y\}$. We say that Y *avoids the cone above* A if $A \not\leq_{\mathrm{T}} Y$. The purpose of this section is to extend, generalize, and simplify the results of Section 4 of Cholak/Greenberg/Miller [7] concerning almost everywhere domination, DNR avoidance, and cone avoidance.

Theorem 5.1. *Given X we can find Y such that $X \leq_{\mathrm{LR}} Y$ and $Y' \leq_{\mathrm{T}} X'$ and* DNR $\not\leq_{\mathrm{w}} \{Y\}$.

Proof. We shall use the following characterization of LR-reducibility.

Lemma 5.2. *Given X, we can find a particular computable measure μ and a particular X-r.e. set I such that the following holds. For all Y, $X \leq_{\mathrm{LR}} Y$ if and only if there exists a Y-r.e. set J such that $\mu(J) < \infty$ and $I \subseteq J$.*

Proof. Let $\mu(I) = \sum_{(n,i)\in I} 1/2^n$ and $I = \{(n,i) \mid n \geq \mathrm{K}^X(i)\}$. Here we are identifying $\mathbb{N}$ with $\mathbb{N} \times \mathbb{N}$. Clearly $\mu(I) \leq 2 < \infty$. Let J be Y-r.e. such that $\mu(J) < \infty$ and $I \subseteq J$. Let c be such that $\mu(J) \leq 2^c$. Then $\sum_{(n,j)\in J} 1/2^{c+n} \leq 1$, so by the Kraft/Chaitin Lemma [42, Corollary 10.6] relative to Y, we can find a prefix-free Y-machine M such that for each $(n,j) \in J$ there exists a bitstring σ such that $|\sigma| = c+n$ and $M(\sigma) = j$. Thus $\mathrm{K}^Y(j) \leq n + O(1)$ for all $(n,j) \in J$. Since $I \subseteq J$, it follows that $\mathrm{K}^Y(j) \leq \mathrm{K}^X(j) + O(1)$ for all j. In other words, $X \leq_{\mathrm{LK}} Y$. Our lemma now follows from Theorem 4.3. □

In order to prove Theorem 5.1 it will suffice to prove the following lemma.

Lemma 5.3. *Let μ be a computable measure. Given X and I such that I is X-r.e. and $\mu(I) < \infty$, we can find Y and J such that J is Y-r.e. and $\mu(J) < \infty$ and $I \subseteq J$ and $Y' \leq_{\mathrm{T}} X'$ and $\mathrm{DNR} \not\leq_{\mathrm{w}} \{Y\}$.*

Proof. Let μ be a computable measure and let I be X-r.e. such that $\mu(I) < \infty$. We define a *forcing condition* to be an ordered pair $p = (\tau^p, a^p)$ where τ^p is a string, a^p is a rational number, and $\mu(I \cup \mathrm{rng}(\tau^p)) < a^p$. Our forcing conditions are partially ordered by letting $q \geq p$ if and only if $\tau^q \supseteq \tau^p$ and $a^q \leq a^p$. Our proof of Lemma 5.3 is based on the following sublemma.

Sublemma 5.4. *Given a forcing condition p, we can find a forcing condition $p^* \geq p$ and a recursively enumerable set T such that*

$$\{\tau^q \mid q \geq p^*\} \ \subseteq\ T \ \subseteq\ \{\tau^q \mid q \geq p\}.$$

Moreover, given p we can find p^* and a recursive index for T using only the oracle X'.

Proof. Given p, let $\epsilon > 0$ be a rational number such that

$$\mu(I \cup \mathrm{rng}(\tau^p)) + \epsilon \ <\ a^p\,.$$

Let a^* be a rational number such that

$$\mu(I \cup \mathrm{rng}(\tau^p)) \ <\ a^* < \mu(I \cup \mathrm{rng}(\tau^p)) + \frac{\epsilon}{2}\,.$$

Let F be a finite subset of I such that $\mu(I \setminus F) < \epsilon/2$. Let $\tau^* = \tau^p{}^\frown\sigma$ where σ is a string such that $\mathrm{rng}(\sigma) = F$. Because I is X-r.e., we can find ϵ and a^* and τ^* using X' as an oracle. Since $I \cup \mathrm{rng}(\tau^*) = I \cup \mathrm{rng}(\tau^p)$, it is clear that $p^* = (\tau^*, a^*)$ is a forcing condition and $p^* \geq p$. Let

$$T = \{\tau \supseteq \tau^* \mid \mu(\mathrm{rng}(\tau) \setminus \mathrm{rng}(\tau^*)) < \epsilon\}\,.$$

Clearly T is r.e. and we can find an r.e. index for T using X' as an oracle. It remains to prove two claims: $T \subseteq \{\tau^q \mid q \geq p\}$ and $\{\tau^q \mid q \geq p^*\} \subseteq T$.

To prove the first claim, assume $\tau \in T$. Then $\tau \supseteq \tau^* \supseteq \tau^p$ and

$$\mu(I \cup \mathrm{rng}(\tau)) \ < \ \mu(I \cup \mathrm{rng}(\tau^*)) + \epsilon \ = \ \mu(I \cup \mathrm{rng}(\tau^p)) + \epsilon \ < \ a^p \,.$$

Thus $q = (\tau, a^p)$ is a forcing condition, and obviously $q \geq p$ and $\tau^q = \tau$.

To prove the second claim, assume $q \geq p^*$. Then $\tau^q \supseteq \tau^*$, hence $\mathrm{rng}(\tau^q) \supseteq \mathrm{rng}(\tau^*) \supseteq F$, hence

$$\mathrm{rng}(\tau^q) \setminus \mathrm{rng}(\tau^*) \ \subseteq \ (I \setminus F) \cup (\mathrm{rng}(\tau^q) \setminus (I \cup \mathrm{rng}(\tau^*))).$$

Since $\mu(I \cup \mathrm{rng}(\tau^q)) < a^q \leq a^* < \mu(I \cup \mathrm{rng}(\tau^*)) + \epsilon/2$ it follows that

$$\mu(\mathrm{rng}(\tau^q) \setminus \mathrm{rng}(\tau^*)) \ \leq \ \frac{\epsilon}{2} + \frac{\epsilon}{2} \ = \ \epsilon$$

hence $\tau^q \in T$. This completes the proof of Sublemma 5.4. □

We now prove Lemma 5.3. We may safely assume that I is nonempty. Starting with any forcing condition p_0, we shall inductively define a sequence of forcing conditions $p_0 \leq p_1 \leq \cdots \leq p_n \leq p_{n+1} \leq \cdots$ where $p_n = (\tau_n, a_n)$. Because $I \neq \emptyset$, it will be easy to arrange that $|\tau_n| \geq n$ for all n. We shall then let $Y = \bigcup_{n=0}^{\infty} \tau_n$ and $J = \mathrm{rng}(Y) = \bigcup_{n=0}^{\infty} \mathrm{rng}(\tau_n)$. It will be easy to arrange that $J \supseteq I$ and $\mu(J) = \inf_n a_n < \infty$. The entire construction will be recursive in X' using Sublemma 5.4. Once the forcing condition p_n is known, we use Sublemma 5.4 to obtain a forcing condition $p_n^* \geq p_n$ and an r.e. set T_n such that $\{\tau^q \mid q \geq p_n^*\} \subseteq T_n \subseteq \{\tau^q \mid q \geq p_n\}$.

In order to insure that $Y' \leq_{\mathrm{T}} X'$, let the forcing condition $p_{2e} = (\tau_{2e}, a_{2e})$ be given. We shall choose $p_{2e+1} \geq p_{2e}$ so as to decide $Y'(e)$. This will insure that $Y' \leq_{\mathrm{T}} X'$, because the entire construction will be $\leq_{\mathrm{T}} X'$.

Case 1: There exists $\tau \in T_{2e}$ such that $\varphi_{e,|\tau|}^{(1),\tau}(e)\downarrow$. In this case, search for such a τ and let $p_{2e+1} = (\tau, a_{2e})$. This is a forcing condition because $\tau \in T_{2e} \subseteq \{\tau^q \mid q \geq p_{2e}\}$. Thus we have forced $Y'(e) = i + 1$ where $\varphi_{e,|\tau|}^{(1),\tau}(e) \downarrow = i$.

Case 2: Not Case 1. In this case let $p_{2e+1} = p_{2e}^*$. It remains to show that p_{2e+1} forces $\varphi_e^{(1),Y}(e)\uparrow$. This follows from the failure of Case 1, because $\{\tau^q \mid q \geq p_{2e+1}\} \subseteq T_{2e}$. Thus we have forced $Y'(e) = 0$.

In order to insure that DNR $\not\leq_{\mathrm{w}} \{Y\}$, let the forcing condition $p_{2e+1} = (\tau_{2e+1}, a_{2e+1})$ be given. We shall choose $p_{2e+2} \geq p_{2e+1}$ to force $\varphi_e^{(1),Y}(n) \downarrow = \varphi_n^{(1)}(n)\downarrow$ or $\varphi_e^{(1),Y}(n)\uparrow$ for some n. Thus $\varphi_e^{(1),Y}$ will not be a DNR function.

Case 1: There exists $\tau \in T_{2e+1}$ such that $\varphi_{e,|\tau|}^{(1),\tau}(n) \downarrow = \varphi_n^{(1)}(n)\downarrow$ for some n. In this case, search for such a τ and let $p_{2e+2} = (\tau, a_{2e+1})$. This is a forcing condition because $\tau \in T_{2e+1} \subseteq \{\tau^q \mid q \geq p_{2e+1}\}$. Thus we have forced $\varphi_e^{(1),Y}(n) \downarrow = \varphi_n^{(1)}(n)\downarrow$.

Case 2: Not Case 1. In this case let $p_{2e+1} = p_{2e}^*$. We claim that p_{2e+1} forces $\varphi_e^{(1),Y}(n)\uparrow$ for some n. Otherwise, for each n search for $\tau \in T_{2e+1}$ such that $\varphi_{e,|\tau|}^{(1),\tau}(n)\downarrow$. Such a τ must exist because $\{\tau^q \mid q \geq p_{2e+2}\} \subseteq T_{2e+1}$. When such a τ is found, define $h(n) = \varphi_{e,|\tau|}^{(1),\tau}(n)$. By the failure of Case 1 we must have either $\varphi_n^{(1)}(n) \uparrow$ or $\varphi_n^{(1)}(n) \downarrow \neq h(n)$. Thus h is a DNR function, but this is impossible because h is recursive.

This completes the proof of Lemma 5.3. □

Finally we prove Theorem 5.1. Let X be given. Let μ and I be as in Lemma 5.2. By Lemma 5.3 let Y and J be such that $\mu(J) < \infty$ and $I \subseteq J$ and $Y' \leq_{\mathrm{T}} X'$ and $\mathrm{DNR} \nleq_{\mathrm{w}} \{Y\}$. From $\mu(J) < \infty$ and $I \subseteq J$ it follows by Lemma 5.2 that $X \leq_{\mathrm{LR}} Y$. This completes the proof of Theorem 5.1. □

We now prove some variants of Theorem 5.1. These variants will be used in Section 7 to build some ω-models which are relevant for the reverse mathematics of measure-theoretic regularity.

Theorem 5.5. *Given X, A, B, C such that $A \nleq_{\mathrm{T}} B$ and $\mathrm{DNR} \nleq_{\mathrm{w}} \{C\}$, we can find Y such that $X \leq_{\mathrm{LR}} Y$ and $A \nleq_{\mathrm{T}} B \oplus Y$ and $\mathrm{DNR} \nleq_{\mathrm{w}} \{C \oplus Y\}$.*

Proof. We imitate the proof of Theorem 5.1. Clearly it suffices to prove the following variant of Lemma 5.3.

Lemma 5.6. *Assume $A \nleq_{\mathrm{T}} B$ and $\mathrm{DNR} \nleq_{\mathrm{w}} \{C\}$. Let μ be a computable measure. Given X and I such that I is X-r.e. and $\mu(I) < \infty$, we can find Y and J such that J is Y-r.e. and $\mu(J) < \infty$ and $I \subseteq J$ and $A \nleq_{\mathrm{T}} B \oplus Y$ and $\mathrm{DNR} \nleq_{\mathrm{w}} \{C \oplus Y\}$.*

Proof. We imitate the proof of Lemma 5.3.

In order to insure that $A \nleq_{\mathrm{T}} B \oplus Y$, let the forcing condition $p_{2e} = (\tau_{2e}, a_{2e})$ be given. We shall choose $p_{2e+1} \geq p_{2e}$ to force $\varphi_e^{(1),B\oplus Y} \neq A$.

Case 1: There exists $\tau \in T_{2e}$ such that $\varphi_{e,|\tau|}^{(1),B\oplus\tau}(n) \downarrow \neq A(n)$ for some n. In this case let $p_{2e+1} = (\tau, a_{2e})$. This is a forcing condition because $\tau \in T_{2e} \subseteq \{\tau^q \mid q \geq p_{2e}\}$. Thus we have forced $\varphi_e^{(1),B\oplus Y}(n) \downarrow \neq A(n)$.

Case 2: Not Case 1. In this case let $p_{2e+1} = p_{2e}^*$. We claim that p_{2e+1} forces $\varphi_e^{(1),B\oplus Y}(n) \uparrow$ for some n. Otherwise, because $\{\tau^q \mid q \geq p_{2e+1}\} \subseteq T_{2e}$, we have that for each n there exists $\tau \in T_{2e}$ such that $\varphi_{e,|\tau|}^{(1),B\oplus\tau}(n) \downarrow$. But for any $\tau \in T_{2e}$ such that $\varphi_{e,|\tau|}^{(1),B\oplus\tau}(n) \downarrow$ we must have $\varphi_{e,|\tau|}^{(1),B\oplus\tau}(n) = A(n)$, by the failure of Case 1. Thus $A \leq_{\mathrm{T}} B$, and this is a contradiction.

In order to insure that $\mathrm{DNR} \nleq_{\mathrm{w}} \{C \oplus Y\}$, let the forcing condition $p_{2e+1} = (\tau_{2e+1}, a_{2e+1})$ be given. We shall choose $p_{2e+2} \geq p_{2e+1}$ so as to force $\varphi_e^{(1),C\oplus Y}$ to be non-DNR.

Case 1: There exists $\tau \in T_{2e+1}$ such that $\varphi_{e,|\tau|}^{(1),C\oplus\tau}(n) \downarrow = \varphi_n^{(1)}(n) \downarrow$ for some n. In this case let $p_{2e+2} = (\tau, a_{2e+1})$. This is a forcing condition because $\tau \in T_{2e+1} \subseteq \{\tau^q \mid q \geq p_{2e+1}\}$. Thus we have forced $\varphi_e^{(1),C\oplus Y}(n) \downarrow = \varphi_n^{(1)}(n) \downarrow$, hence $\varphi_e^{(1),C\oplus Y}$ is not DNR.

Case 2: Not Case 1. In this case let $p_{2e+2} = p_{2e+1}^*$. We claim that p_{2e+2} forces $\varphi_e^{(1),C\oplus Y}(n) \uparrow$ for some n. Otherwise, for each n search for $\tau \in T_{2e+1}$ such that $\varphi_{e,|\tau|}^{(1),C\oplus\tau}(n) \downarrow$. Such a τ must exist because $\{\tau^q \mid q \geq p_{2e+2}\} \subseteq T_{2e+1}$.

When such a τ is found, define $h(n) = \varphi_{e,|\tau|}^{(1),C\oplus\tau}(n)$. By the failure of Case 1 we must have either $\varphi_n^{(1)}(n)\uparrow$ or $\varphi_n^{(1)}(n)\downarrow\neq h(n)$. Thus h is a DNR function, but this is impossible because $h \leq_{\mathrm{T}} C$ and $\mathrm{DNR} \nleq_{\mathrm{w}} \{C\}$. □

The proof of Theorem 5.5 is now complete. □

We now generalize Theorem 5.5 replacing the oracles A, B, C by a countable sequence of oracles A_i, B_i, C_i where $i = 0, 1, 2, \ldots$.

Theorem 5.7. *Given X and $A_i \nleq_{\mathrm{T}} B_i$ and $\mathrm{DNR} \nleq_{\mathrm{w}} \{C_i\}$ for all i, we can find Y such that $X \leq_{\mathrm{LR}} Y$ and $A_i \nleq_{\mathrm{T}} B_i \oplus Y$ and $\mathrm{DNR} \nleq_{\mathrm{w}} \{C_i \oplus Y\}$ for all i.*

Proof. This is a routine generalization of Theorem 5.5. We omit the details. □

Theorem 5.8. *Given X and $A_i \nleq_{\mathrm{T}} B_i$ and $\mathrm{DNR} \nleq_{\mathrm{w}} \{C_i\}$ and $\mathrm{PA} \nleq_{\mathrm{w}} \{D_i\}$ for all i, we can find Y such that $X \leq_{\mathrm{LR}} Y$ and $A_i \nleq_{\mathrm{T}} B_i \oplus Y$ and $\mathrm{DNR} \nleq_{\mathrm{w}} \{C_i \oplus Y\}$ and $\mathrm{PA} \nleq_{\mathrm{w}} \{D_i \oplus Y\}$ for all i.*

Proof. This is a variant[1] of Theorem 5.7 with PA instead of DNR. The proof is similar, using $\mathrm{DNR} \cap 2^{\mathbb{N}}$ instead of DNR and recalling the well known fact (see for instance [41]) that $\mathrm{DNR} \cap 2^{\mathbb{N}}$ is Muchnik equivalent to PA. □

Let us say that X *is arithmetical in* Y, abbreviated $X \leq_{\mathrm{a}} Y$, if $X \leq_{\mathrm{T}} Y^{(n)}$ for some n.

Theorem 5.9. *Given X and $A_i \nleq_{\mathrm{a}} B_i$ for all i, we can find Y such that $X \leq_{\mathrm{LR}} Y$ and $A_i \nleq_{\mathrm{a}} B_i \oplus Y$ for all i.*

Proof. This is like Theorem 5.7 replacing $\leq_{\mathrm{T}}$ by $\leq_{\mathrm{a}}$. The proof is similar. □

6 Mass problems

For each recursive ordinal α, let B_α be the set of Turing oracles with the properties listed in Corollary 4.12. In other words, $\mathrm{B}_\alpha = \{Y \mid 0^{(\alpha)} \leq_{\mathrm{LR}} Y\}$ or equivalently

$$\mathrm{B}_\alpha = \{Y \mid \text{every } \Sigma^0_{\alpha+2} \text{ set includes a } \Sigma^{0,Y}_2 \text{ set of the same measure}\}.$$

We may view B_α as a mass problem. Clearly these mass problems are of interest from the viewpoint of reverse mathematics, specifically the reverse mathematics of measure-theoretic regularity.

In this section we shall prove that B_α is Σ^0_3. From this it will follow that the Muchnik degrees $\mathbf{b}_\alpha = \deg_{\mathrm{w}}(\mathrm{B}_\alpha) \in \mathcal{D}_{\mathrm{w}}$ embed nicely into $\mathcal{E}_{\mathrm{w}}$. See also Figure 1 below.

Definition 6.1. If S is a set of Turing oracles, we write

$$S^{\mathrm{LR}} = \{Y \mid \exists X\, (X \in S \text{ and } X \leq_{\mathrm{LR}} Y)\}.$$

[1]Obviously many other such variants also hold.

Thus S^{LR} is the upward closure of S under LR-reducibility.

Theorem 6.2. *If S is Σ^0_3 then S^{LR} is again Σ^0_3.*

Proof. We need the following lemma.

Lemma 6.3. *The 2-place predicate $X \leq_{\mathrm{LR}} Y$ is $\Sigma^{0,X',Y'}_2$. In other words, we have a Σ^0_2 predicate $L(U,V)$ such that for all X and Y, $X \leq_{\mathrm{LR}} Y$ if and only if $L(X',Y')$ holds.*

Proof. Let $I^X_c = \{(n,i) \mid n \geq \mathrm{K}^X(i) + c\}$. Clearly the sets I^X_c for $c = 0,1,2,\ldots$ are uniformly $\Sigma^{0,X}_1$. It is also clear that $X \leq_{\mathrm{LK}} Y$ if and only if $\exists c\,(I^X_c \subseteq I^Y_0)$, i.e., $\exists c\,\forall n\,\forall i\,((n,i) \in I^X_c \Rightarrow (n,i) \in I^Y_0)$. Thus $X \leq_{\mathrm{LK}} Y$ is $\Sigma^{0,X',Y'}_2$. It follows by Theorem 4.3 that $X \leq_{\mathrm{LR}} Y$ is also $\Sigma^{0,X',Y'}_2$, Q.E.D. □

Since S is Σ^0_3, let $R(X,k,m,n)$ be a recursive predicate such that $S = \{X \mid \exists k\,\forall m\,\exists n\,R(X,k,m,n)\}$. Let $P = \{\langle k\rangle^\frown X \oplus f \mid \forall m\,(f(m) = \text{the least } n \text{ such that } R(X,k,m,n))\}$. Clearly P is Π^0_1 and each $X \in S$ is Turing equivalent to some $\langle k\rangle^\frown X \oplus f \in P$ and vice versa. Thus, replacing S by P, we may safely assume that S is a Π^0_1 subset of $\mathbb{N}^\mathbb{N}$.

Since S is Π^0_1, it follows by Lemma 2.8 that $S' = \{X' \mid X \in S\}$ is again Π^0_1. Let $L(U,V)$ be a Σ^0_2 predicate as in Lemma 6.3. Thus $Y \in S^{\mathrm{LR}}$ if and only if $L(Z,Y')$ holds for some $Z \in S'$. Since $L(U,V)$ is Σ^0_2, let $Q(j,U,V)$ be a Π^0_1 predicate such that $L(U,V) \equiv \exists j\,Q(j,U,V)$.

Recall also Lemma 4.5 which says that every $X \leq_{\mathrm{LR}} Y$ is jump-traceable by Y with bounding function 2^{c+n} for some constant c. Given e, c, n and Y, let $F^Y_{e,c,n}$ be a finite set defined as follows. If $\varphi^{(1)}_e(n)\uparrow$ let $F^Y_{e,c,n} = \emptyset$. Otherwise we have $\varphi^{(1)}_e(n)\downarrow= i$ for some i, so let $F^Y_{e,c,n}$ consist of W^Y_i enumerated so long as its cardinality is $\leq 2^{c+n}$. More precisely $F^Y_{e,c,n}$ is the union of the sets $W^Y_{i,s}$ over all s such that $|W^Y_{i,s}| \leq 2^{c+n}$. Clearly the sets $F^Y_{e,c,n}$ are finite and uniformly Y-r.e. It follows that this sequence of finite sets is canonically Y'-computable. Moreover, Lemma 4.5 tells us that for every $X \leq_{\mathrm{LR}} Y$ there exist constants e and c such that $X'(n) \in F^Y_{e,c,n}$ for all n.

Combining the last two paragraphs, we see that $Y \in S^{\mathrm{LR}}$ if and only if there exist j,e,c,Z such that $Z \in S'$ and $Q(j,Z,Y')$ and $\forall n\,(Z(n) \in F^Y_{e,c,n})$. Thus $S^{\mathrm{LR}} = \{Y \mid \exists j\,\exists e\,\exists c\,(P^Y_{j,e,c} \neq \emptyset)\}$ where

$$P^Y_{j,e,c} = \{Z \mid Z \in S' \text{ and } Q(j,Z,Y') \text{ and } \forall n\,(Z(n) \in F^Y_{e,c,n})\}$$

is $\Pi^{0,Y'}_1$ and Y'-recursively bounded uniformly in j,e,c. From this it follows by compactness that $\{(j,e,c) \mid P^Y_{j,e,c} \neq \emptyset\}$ is $\Pi^{0,Y'}_1$ uniformly in Y'. Hence $\{Y' \mid Y \in S^{\mathrm{LR}}\}$ is Σ^0_2 and thus S^{LR} is Σ^0_3.

This completes the proof of Theorem 6.2. □

The next lemma shows how to embed a large part of $\mathcal{D}_{\mathrm{w}}$ into $\mathcal{E}_{\mathrm{w}}$.

Lemma 6.4. *If $\mathbf{s} = \deg_{\mathrm{w}}(S)$ where S is Σ^0_3, then $\inf(\mathbf{s},\mathbf{1})$ belongs to $\mathcal{E}_{\mathrm{w}}$.*

Proof. This is a consequence of the Embedding Lemma of Simpson [43, Lemma 3.6]. See also our exposition in [46]. ☐

Definition 6.5. For each recursive ordinal α let $\mathrm{B}_\alpha = \{Y \mid 0^{(\alpha)} \leq_{\mathrm{LR}} Y\}$ and consider the Muchnik degree $\mathbf{b}_\alpha = \deg_{\mathrm{w}}(\mathrm{B}_\alpha)$. Consider also the Muchnik degrees

$$\mathbf{1} = \deg_{\mathrm{w}}(\{X \mid X \text{ is a completion of Peano Arithmetic}\})$$

and

$$\mathbf{r} = \deg_{\mathrm{w}}(\{X \mid X \text{ is random in the sense of Martin-Löf}\})$$

and

$$\mathbf{d} = \deg_{\mathrm{w}}(\{X \mid X \text{ is diagonally nonrecursive}\})$$

and

$$\mathbf{0} = \deg_{\mathrm{w}}(\{X \mid X \text{ is recursive}\}).$$

It is well known that $\mathbf{1}$ and $\mathbf{0}$ are the top and bottom degrees in $\mathcal{E}_{\mathrm{w}}$. By [41, 43] we know that the degrees $\mathbf{d}$ and $\mathbf{r}$ also belong to $\mathcal{E}_{\mathrm{w}}$ and that $\mathbf{0} < \mathbf{d} < \mathbf{r} < \mathbf{1}$.

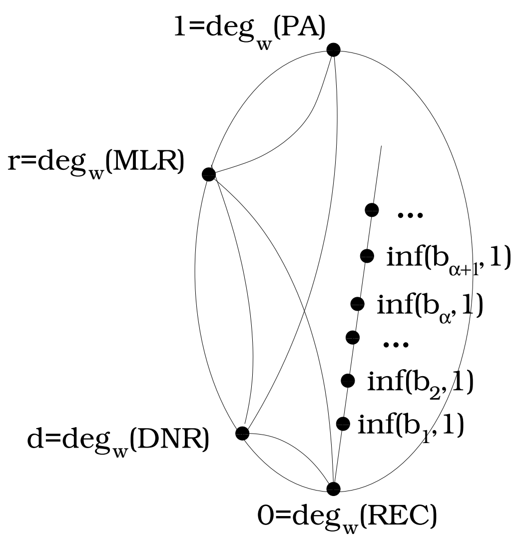

Figure 1: A picture of $\mathcal{E}_{\mathrm{w}}$. Each of the black dots represents a specific, natural Muchnik degree in $\mathcal{E}_{\mathrm{w}}$. Namely $\mathbf{1} = \deg_{\mathrm{w}}(\{X \mid X$ is a completion of Peano Arithmetic$\})$, $\mathbf{r} = \deg_{\mathrm{w}}(\{X \mid X$ is Martin-Löf random$\})$, $\mathbf{d} = \deg_{\mathrm{w}}(\{X \mid X$ is diagonally nonrecursive$\})$, $\mathbf{0} = \deg_{\mathrm{w}}(\{X \mid X$ is recursive$\})$, and $\mathbf{b}_\alpha = \deg_{\mathrm{w}}(\{Y \mid 0^{(\alpha)} \leq_{\mathrm{LR}} Y\})$ for each $\alpha < \omega_1^{\mathrm{CK}}$.

Theorem 6.6. *For each $\alpha < \omega_1^{\mathrm{CK}}$ the mass problem B_α is Σ^0_3. Hence $\inf(\mathbf{b}_\alpha, \mathbf{1})$ belongs to $\mathcal{E}_{\mathrm{w}}$. Moreover, for all $\alpha < \beta < \omega_1^{\mathrm{CK}}$ we have $\inf(\mathbf{b}_\alpha, \mathbf{1}) < \inf(\mathbf{b}_\beta, \mathbf{1})$. Also, if $\alpha > 0$ then $\inf(\mathbf{b}_\alpha, \mathbf{1})$ is incomparable with $\mathbf{d}$ and $\mathbf{r}$.*

Proof. Fix a recursive ordinal α. From hyperarithmetical theory it is well known that $0^{(\alpha)}$ is a Σ^0_3 singleton. (References are in the proof of Theorem 4.9 above.) In other words, the one-element set $\{0^{(\alpha)}\}$ is Σ^0_3. It follows by Theorem 6.2 that $\mathrm{B}_\alpha = \{0^{(\alpha)}\}^{\mathrm{LR}}$ is Σ^0_3. Hence by Lemma 6.4 we have $\inf(\mathbf{b}_\alpha, \mathbf{1}) \in \mathcal{E}_{\mathrm{w}}$.

Trivially $\mathbf{b}_\alpha \leq \mathbf{b}_\beta$ for all $\alpha < \beta < \omega_1^{\mathrm{CK}}$. Therefore, to prove $\inf(\mathbf{b}_\alpha, \mathbf{1}) < \inf(\mathbf{b}_\beta, \mathbf{1})$ it suffices to prove $\inf(\mathbf{b}_{\alpha+1}, \mathbf{1}) \not\leq \mathbf{b}_\alpha$. By Theorem 5.1 let Y be such that $0^{(\alpha)} \leq_{\mathrm{LR}} Y$ and $Y' \leq_{\mathrm{T}} 0^{(\alpha+1)}$ and $\mathrm{DNR} \not\leq_{\mathrm{w}} \{Y\}$. Since $0^{(\alpha+2)} \not\leq_{\mathrm{T}} Y'$, it follows by Corollary 4.10 that $0^{(\alpha+1)} \not\leq_{\mathrm{LR}} Y$. Thus $\mathrm{B}_{\alpha+1} \cup \mathrm{DNR} \not\leq_{\mathrm{w}} \{Y\}$, and this implies that $\inf(\mathbf{b}_{\alpha+1}, \mathbf{d}) \not\leq \mathbf{b}_\alpha$. From this we clearly have $\inf(\mathbf{b}_{\alpha+1}, \mathbf{1}) \not\leq \mathbf{b}_\alpha$ and $\mathbf{d} \not\leq \inf(\mathbf{b}_\alpha, \mathbf{1})$.

By the Low Basis Theorem (see for instance [41]) let Z be Martin-Löf random and *low*, i.e., $Z' \leq_{\mathrm{T}} 0'$. By Corollary 4.10 (see also [42]) we know that each $Y \in \mathrm{B}_\alpha$ for $\alpha > 0$ is *high*, i.e., $0'' \leq_{\mathrm{T}} Y'$. Thus $\mathrm{B}_\alpha \not\leq_{\mathrm{w}} \{Z\}$. Moreover, in view of Stephan's Theorem [53] (see also our exposition in [44, Section 6]) we have $\mathrm{PA} \not\leq_{\mathrm{w}} \{Z\}$. Thus $\mathrm{B}_\alpha \cup \mathrm{PA} \not\leq_{\mathrm{w}} \{Z\}$, and this implies that $\inf(\mathbf{b}_\alpha, \mathbf{1}) \not\leq \mathbf{r}$.

For $\alpha > 0$ we have seen that $\inf(\mathbf{b}_\alpha, \mathbf{1})$ is $\not\geq \mathbf{d}$ and $\not\leq \mathbf{r}$. From this plus $\mathbf{d} < \mathbf{r}$ it follows that $\inf(\mathbf{b}_\alpha, \mathbf{1})$ is incomparable with $\mathbf{d}$ and $\mathbf{r}$, Q.E.D. $\square$

Remark 6.7. The inequalities which were stated in Theorem 6.6 are illustrated in Figure 1. A more precise account is given in the next theorem.

Theorem 6.8. *For $0 < \alpha < \beta < \omega_1^{\mathrm{CK}}$ we have*

$$\mathbf{0} < \inf(\mathbf{d}, \inf(\mathbf{b}_\alpha, \mathbf{1})) < \inf(\mathbf{d}, \inf(\mathbf{b}_\beta, \mathbf{1})) < \mathbf{d} \tag{1}$$

and

$$\mathbf{r} < \sup(\mathbf{r}, \inf(\mathbf{b}_\alpha, \mathbf{1})) < \sup(\mathbf{r}, \inf(\mathbf{b}_\beta, \mathbf{1})) < \mathbf{1}. \tag{2}$$

Proof. The inequalities (1) are already clear from the proof of Theorem 6.6.

By a theorem of Nies (see our exposition in [44, Corollary 5.4]), given $X \geq_{\mathrm{T}} 0'$ we can find $Z \leq_{\mathrm{T}} X$ such that $0' \not\leq_{\mathrm{T}} Z$ and $X \leq_{\mathrm{LR}} Z$ and Z is random in the sense of Martin-Löf. Letting $X = 0^{(\alpha)}$ with $\alpha > 0$, we obtain a random Z such that $0^{(\alpha)} \leq_{\mathrm{LR}} Z$ and $0^{(\alpha+1)} \not\leq_{\mathrm{LR}} Z$ and $0' \not\leq_{\mathrm{T}} Z$. Since Z is random and $0' \not\leq_{\mathrm{T}} Z$, it follows by Stephan's Theorem that $\mathrm{PA} \not\leq_{\mathrm{w}} \{Z\}$. Thus $\inf(\mathbf{b}_{\alpha+1}, \mathbf{1}) \not\leq \sup(\mathbf{r}, \mathbf{b}_\alpha)$, and (2) follows easily from this. $\square$

Remark 6.9. For each $\alpha < \omega_1^{\mathrm{CK}}$ let $\mathbf{c}_\alpha = \deg_{\mathrm{w}}(C_\alpha)$ where

$$C_\alpha = \{Z \mid 0^{(\alpha)} \leq_{\mathrm{LR}} Z \text{ and } Z \text{ is random in the sense of Martin-Löf}\}.$$

As above we have $\inf(\mathbf{c}_\alpha, \mathbf{1}) \in \mathcal{E}_{\mathrm{w}}$ and $\mathbf{r} < \inf(\mathbf{c}_\alpha, \mathbf{1}) < \inf(\mathbf{c}_\beta, \mathbf{1}) < \mathbf{1}$ whenever $0 < \alpha < \beta < \omega_1^{\mathrm{CK}}$. See also the additional information in Simpson [44].

7 Consequences for reverse mathematics

Remark 7.1. As noted in [40, 47, Theorem VIII.3.15], the principal axiom of ATR_0 is equivalent to the existence of $X^{(\alpha)}$ for all X and all $\alpha < \omega_1^X$. Therefore, by formalizing Theorem 3.3 within ATR_0, we see that ATR_0 suffices to prove measure-theoretic regularity at all levels of the Borel hierarchy. See also the discussion of Borel sets in ATR_0 in [40, 47, Section V.3]. This raises the question:

> Which set existence axioms are needed in order to prove measure-theoretic regularity in the Borel hierarchy?

This is a typical question of reverse mathematics. We shall now apply our results on Muchnik degrees in order to build some ω-models (see [40, 47, Chapter VIII]) which are relevant for this question.

Definition 7.2 (MTR-models)**.** Let M be an ω-model of RCA_0. For $S \subseteq 2^{\mathbb{N}}$ we say that S is *M-Borel* if S is $\Sigma^{0,X}_\alpha$ for some $X \in M$ and some $\alpha < \omega_1^X$. We say that S is *M-F_σ* if S is $\Sigma^{0,Y}_2$ for some $Y \in M$. We say that M is an MTR-*model* if every M-Borel set includes an M-F_σ set of the same measure. The acronym MTR stands for "measure-theoretic regularity."

Lemma 7.3. *Let M be an ω-model of RCA_0. Then M is an* MTR*-model if and only if*

$$(\forall X \in M)\,(\forall \alpha < \omega_1^X)\,(\exists Y \in M)\,(X^{(\alpha)} \leq_{\mathrm{LR}} Y). \tag{3}$$

Proof. Since M is closed under the pairing function $\oplus$, property (3) easily implies the stronger-looking property

$$(\forall X \in M)\,(\forall \alpha < \omega_1^X)\,(\exists Y \in M)\,(X^{(\alpha)} \leq_{\mathrm{LR}} Y \text{ and } X \leq_{\mathrm{T}} Y).$$

But then, by $1 \Rightarrow 5$ of Theorem 4.11, M is an MTR-model. Conversely, assume that M is an MTR-model and let $X \in M$ and $\alpha < \omega_1^X$ be given. Consider a universal $\Sigma^{0,X}_{\alpha+2}$ set S defined by

$$S = \{\langle \underbrace{0,\ldots,0}_{n}, 1\rangle^\frown Z \mid n \in \mathbb{N}, Z \in S_n\}$$

where S_n, $n \in \mathbb{N}$ is a recursive enumeration of the $\Sigma^{0,X}_{\alpha+2}$ sets. Since M is an MTR-model, let $Y \in M$ be such that S includes a $\Sigma^{0,Y}_2$ set of the same measure. From the universality of S it follows that every $\Sigma^{0,X}_{\alpha+2}$ set includes a $\Sigma^{0,Y}_2$ set of the same measure. But then by $5 \Rightarrow 1$ of Theorem 4.11 we have $X^{(\alpha)} \leq_{\mathrm{LR}} Y$. This completes the proof. □

Theorem 7.4. *We can find* MTR*-models M_1, M_2, M_3, M_4 satisfying* $\mathsf{RCA}_0 + \neg\,\mathsf{WWKL}_0$ *and* $\mathsf{WWKL}_0 + \neg\,\mathsf{WKL}_0$ *and* $\mathsf{WKL}_0 + \neg\,\mathsf{ACA}_0$ *and* $\mathsf{ACA}_0 + \neg\,\mathsf{ATR}_0$ *respectively. Moreover, given a sequence of oracles A_i such that $\forall i\,(A_i \not\leq_{\mathrm{T}} 0)$, we can arrange that $A_i \notin M_j$ for all $i \in \mathbb{N}$ and $j = 1, 2, 3$. The same holds for $j = 4$ provided $\forall i\, \forall n\,(A_i \not\leq_{\mathrm{T}} 0^{(n)})$.*

Proof. Let M be a countable ω-model of ACA_0 [40, 47, Chapter VIII] which is closed under relative hyperarithmeticity, i.e., $(\forall X \in M)\,(\forall \alpha < \omega_1^X)\,(X^{(\alpha)} \in M)$. Assume also that $\bigoplus_{i=0}^{\infty} A_i \in M$. We shall build M_j as a submodel of M with the following property:

$$(\forall X \in M)\,(\exists Y \in M_j)\,(X \leq_{\mathrm{LR}} Y).$$

By Lemma 7.3 this insures that M_j is an MTR-model.

Let X_n for $n = 0, 1, 2, \ldots$ be a fixed enumeration of M.

To build M_1 start with $\forall i\,(A_i \not\leq_{\mathrm{T}} 0)$ and apply Theorem 5.7 repeatedly for $n = 0, 1, 2, \ldots$ to obtain $Y_n \in M$ such that $X_n \leq_{\mathrm{LR}} Y_n$ and $\mathrm{DNR} \not\leq_{\mathrm{w}} \{Y_n\}$ and $\forall i\,(A_i \not\leq_{\mathrm{T}} Y_n)$ and $Y_n \leq_{\mathrm{T}} Y_{n+1}$. Letting $M_1 = \{Y \mid \exists n\,(Y \leq_{\mathrm{T}} Y_n)\}$ we have $M_1 \cap \mathrm{DNR} = \emptyset$ and $\forall i\,(A_i \notin M_1)$. Since $M_1 \cap \mathrm{DNR} = \emptyset$, there is no $Z \in M_1$ which is Martin-Löf random. In particular $M_1 \models \mathsf{RCA}_0 + \neg\,\mathsf{WWKL}_0$ as required.

For M_2 we need a lemma:

Lemma 7.5. *Given X and $A_i \not\leq_{\mathrm{T}} B_i$ and $\mathrm{PA} \not\leq_{\mathrm{w}} \{C_i\}$ for all i, we can find Z such that Z is Martin-Löf random relative to X and $A_i \not\leq_{\mathrm{T}} B_i \oplus Z$ and $\mathrm{PA} \not\leq_{\mathrm{w}} \{C_i \oplus Z\}$ for all i.*

Proof. For any X the set $\{Z \mid Z$ is Martin-Löf random relative to $X\}$ is of measure 1. Also, $A_i \not\leq_{\mathrm{T}} B_i$ implies that $\{Z \mid A_i \not\leq_{\mathrm{T}} B_i \oplus Z\}$ is of measure 1, and $\mathrm{PA} \not\leq_{\mathrm{w}} \{C_i\}$ implies that $\{Z \mid \mathrm{PA} \not\leq_{\mathrm{w}} \{C_i \oplus Z\}\}$ is of measure 1. Letting Z belong to the intersection of these sets of measure 1, we have our lemma. ☐

To build M_2 start with $\forall i\,(A_i \not\leq_{\mathrm{T}} 0)$ and apply Theorem 5.8 and Lemma 7.5 repeatedly for $n = 0, 1, 2, \ldots$ to obtain $Y_n \in M$ and $Z_n \in M$ such that $X_n \leq_{\mathrm{LR}} Y_n$ and $\mathrm{PA} \not\leq_{\mathrm{w}} \{Y_n\}$ and $\forall i\,(A_i \not\leq_{\mathrm{T}} Y_n)$ and Z_n is Martin-Löf random[2] relative to $X_n \oplus Y_n$ and $\mathrm{PA} \not\leq_{\mathrm{w}} \{Y_n \oplus Z_n\}$ and $\forall i\,(A_i \not\leq_{\mathrm{T}} Y_n \oplus Z_n)$ and $Y_n \oplus Z_n \leq_{\mathrm{T}} Y_{n+1}$. Letting $M_2 = \{Y \mid \exists n\,(Y \leq_{\mathrm{T}} Y_n)\}$ we have $M_2 \cap \mathrm{PA} = \emptyset$ and $\forall i\,(A_i \notin M_2)$ and

$$(\forall X \in M)\,(\exists Z \in M_2)\,(Z \text{ is Martin-Löf random relative to } X).$$

In particular $M_2 \models \mathsf{WWKL}_0 + \neg\,\mathsf{WKL}_0$ as required.

For M_3 we need another lemma:

Lemma 7.6. *Given Y and $A_i \not\leq_{\mathrm{T}} Y$ for all i, we can find Z such that Z is* PA*-complete over Y and $A_i \not\leq_{\mathrm{T}} Y \oplus Z$ for all i.*

Proof. This is the Gandy/Kreisel/Tait Theorem [14]. See also our exposition in [40, 47, Theorem VIII.2.2.4]. ☐

To build M_3 start with $\forall i\,(A_i \not\leq_{\mathrm{T}} 0)$ and apply Theorem 5.8 and Lemma 7.6 repeatedly for $n = 0, 1, 2, \ldots$ to obtain $Y_n \in M$ and $Z_n \in M$ such that $X_n \leq_{\mathrm{LR}} Y_n$ and $\forall i\,(A_i \not\leq_{\mathrm{T}} Y_n)$ and Z_n is PA-complete over Y_n and $\forall i\,(A_i \not\leq_{\mathrm{T}} Y_n \oplus Z_n)$ and $Y_n \oplus Z_n \leq_{\mathrm{T}} Y_{n+1}$. Letting $M_3 = \{Y \mid \exists n\,(Y \leq_{\mathrm{T}} Y_n)\}$ we have $\forall i\,(A_i \notin M_3)$ and

[2] Clearly we can replace Martin-Löf randomness by much stronger randomness notions.

$$(\forall Y \in M_3)\,(\exists Z \in M_3)\,(Z \text{ is PA-complete over } Y).$$

Letting $A_0 = 0'$ we have $0' \notin M_3$ so $M_3 \models \mathsf{WKL}_0 + \neg\,\mathsf{ACA}_0$ as required.

To build M_4 start with $\forall i\,(A_i \not\leq_{\mathrm{a}} 0)$ and apply Theorem 5.9 repeatedly for $n = 0, 1, 2, \ldots$ to obtain $Y_n \in M$ such that $X_n \leq_{\mathrm{LR}} Y_n$ and $\forall i\,(A_i \not\leq_{\mathrm{a}} Y_n)$ and $Y_n' \leq_{\mathrm{T}} Y_{n+1}$. Letting $M_4 = \{Y \mid \exists n\,(Y \leq_{\mathrm{T}} Y_n)\}$ we have $\forall i\,(A_i \notin M_4)$ and $(\forall Y \in M_4)\,(Y' \in M_4)$. Letting $A_0 = 0^{(\omega)}$ we have $0^{(\omega)} \notin M_4$ so $M_4 \models \mathsf{ACA}_0 + \neg\,\mathsf{ATR}_0$ as required. □

Remark 7.7. In the proof of Theorem 7.4, we can insert extra steps into the construction to insure that

$$M = \{X \mid (\exists Y \in M_j)\,(X \leq_{\mathrm{T}} Y')\}$$

for $j = 1, 2, 3$ and

$$M = \{X \mid (\exists Y \in M_j)\,(X \leq_{\mathrm{T}} Y^{(\omega)})\}$$

for $j = 4$. Namely, for $j = 1, 2, 3$ and $n = 0, 1, 2, \ldots$ we can arrange that G_n is 1-generic relative to Y_n and $X_n \leq_{\mathrm{T}} (Y_n \oplus G_n)'$ and $G_n \leq_{\mathrm{T}} Y_{n+1}$. For $j = 4$ we can arrange that G_n is ω-generic relative to Y_n and $X_n \leq_{\mathrm{T}} (Y_n \oplus G_n)^{(\omega)}$ and $G_n \leq_{\mathrm{T}} Y_{n+1}$. By Cole/Simpson [9, Section 3] these extra steps are compatible with the other requirements of the construction.

Remark 7.8. Let M and M_j be as in Remark 7.7. Then clearly M is interpretable in M_j. Moreover, if M is an ω-model of ATR_0 then M_j satisfies measure-theoretic regularity for all levels of the Borel hierarchy along countable well-orderings with a sufficient amount of transfinite induction.

Remark 7.9. Our results above are stated for ω-models. However, as usual in reverse mathematics, we can extend our results to non-ω-models by formalizing our recursion-theoretic arguments within appropriate subsystems of second-order arithmetic.

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