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A Dual Form of Ramsey's Theorem

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Let $k \in \omega$, where ω is the set of all natural numbers. Ramsey's Theorem deals with colorings of the k-element subsets of ω. Our dual form deals with colorings of the k-element partitions of ω. Let $(\omega)^k$ (respectively $(\omega)^\omega$) be the set of all partitions of ω having exactly k (respectively infinitely many) blocks. Given $X \in (\omega)^\omega$ let $(X)^k$ be the set of all $Y \in (\omega)^k$ such that Y is coarser than X. *Dual Ramsey Theorem.* If $(\omega)^k = C_0 \cup \cdots \cup C_{l-1}$ where each C_i is Borel then there exists $X \in (\omega)^\omega$ such that $(X)^k \subseteq C_i$ for some $i < l$. *Dual Galvin–Prikry Theorem.* Same as before with k replaced by ω. We also obtain dual forms of theorems of Ellentuck and Mathias. Our results also provide an infinitary generalization of the Graham–Rothschild "parameter set" theorem [*Trans. Amer. Math. Soc.* **159** (1971), 257–292] and a new proof of the Halpern–Läuchli Theorem [*Trans. Amer. Math. Soc.* **124** (1966), 360–367].

1. INTRODUCTION

The purpose of this paper is to establish a combinatorial theorem which is in a certain sense the dual of Ramsey's Theorem. The original theorem of Ramsey is concerned with colorings of the k-element subsets of a fixed infinite set. Our dual form is concerned with colorings of the k-element partitions of a fixed infinite set.

We begin by recalling Ramsey's Theorem [32]. Let ω be the set of natural numbers. Ramsey's Theorem says that if the k-element subsets of ω are colored with finitely many colors, then there exists an infinite subset of ω all of whose k-element subsets have the same color. In order to state Ramsey's

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Theorem more precisely, we introduce some notation. Let $[\omega]^\omega$ be the set of all infinite subsets of ω. For $k \in \omega$ and $X \in [\omega]^\omega$ let $[X]^k$ be the set of all k-element subsets of X. With this notation we have:

1.1. RAMSEY'S THEOREM. *If $[\omega]^k = C_0 \cup \cdots \cup C_{l-1}$ then there exists $X \in [\omega]^\omega$ such that $[X]^k \subseteq C_i$ for some i.*

We now state our dual form of Ramsey's Theorem. By a *partition of* ω we mean a collection of pairwise disjoint, nonempty subsets of ω whose union is all of ω. The elements of a partition of ω are called its *blocks*. An *infinite partition of* ω is a partition of ω having infinitely many blocks. A *k-element partition of* ω is a partition of ω having exactly k blocks. If X and Y are partitions of ω, we say that Y *is coarser than* X if each block of X is a subset of some block of Y. The dual form of Ramsey's Theorem reads as follows: if the k-element partitions of ω are colored in a "nice" way with finitely many colors, then there exists an infinite partition of ω such that all coarser k-element partitions of ω have the same color.

In order to state our dual form of Ramsey's Theorem more precisely, we introduce some more notation. Let $(\omega)^\omega$ be the set of all infinite partitions of ω. For $k \in \omega$ let $(\omega)^k$ be the set of all k-element partitions of ω. For $X \in (\omega)^\omega$ let $(X)^k$ be the set of all $Y \in (\omega)^k$ such that Y is coarser than X. If Y is any partition of ω, we may identify Y with a binary relation $R_Y \subseteq \omega \times \omega$, where $(m, n) \in R_Y$ if and only if m and n belong to the same block of Y. The set of all binary relations, $\{\text{true}, \text{false}\}^{\omega \times \omega}$, is a topological space, where $\{\text{true}, \text{false}\}$ is endowed with the discrete topology. Thus $(\omega)^k$ and $(\omega)^\omega$ become topological spaces under the topology inherited from the space of binary relations. We call a subset of $(\omega)^k$ or $(\omega)^\omega$ "nice" if it is a Borel set, i.e., it belongs to the σ-algebra generated by the open sets of the appropriate topology. With this understanding we have:

1.2. DUAL RAMSEY THEOREM. *If $(\omega)^k = C_0 \cup \cdots \cup C_{l-1}$, where each C_i is Borel, then there exists $X \in (\omega)^\omega$ such that $(X)^k \subseteq C_i$ for some i.*

Dual Ramsey Theorem 1.2 will be proved in Section 2 except for a lemma whose proof will be postponed until Section 6.

In Section 4 we shall go on to obtain an "infinite exponent" version of Theorem 1.2. This is a dual form of the Galvin–Prikry Theorem [10]. For $X \in (\omega)^\omega$ let $(X)^\omega$ be the set of all $Y \in (\omega)^\omega$ such that Y is coarser than X. Then we have:

1.3. DUAL GALVIN–PRIKRY THEOREM. *If $(\omega)^\omega = C_0 \cup \cdots \cup C_{l-1}$, where each C_i is Borel, then there exists $X \in (\omega)^\omega$ such that $(X)^\omega \subseteq C_i$ for some i.*

Besides proving Theorems 1.2 and 1.3, we shall also explore the extent to which the hypothesis "each C_i Borel" in Theorems 1.2 and 1.3 can be weakened. That this hypothesis cannot be dropped entirely is shown by the following counterexample.

1.4. COUNTEREXAMPLE. *There exists a coloring $(\omega)^2 = C_0 \cup C_1$ such that for all $X \in (\omega)^\omega$ neither $(X)^2 \subseteq C_0$ nor $(X)^2 \subseteq C_1$.*

To see this, let $(\omega)^\omega = \{X_\alpha : \alpha < 2^{\aleph_0}\}$ be a well ordered list of all the infinite partitions of ω. We construct C_0 and C_1 by transfinite induction. At stage 0 put $C_0^0 = C_1^0 = \varnothing$. At stage $\alpha + 1$ note inductively that $|C_0^\alpha \cup C_1^\alpha| < 2^{\aleph_0}$ so we can choose Y_0^α, $Y_1^\alpha \in (X_\alpha)^2 \setminus (C_0^\alpha \cup C_1^\alpha)$ such that $Y_0^\alpha \neq Y_1^\alpha$. Put $C_i^{\alpha+1} = C_i^\alpha \cup \{Y_i^\alpha\}$, $i = 0, 1$. At limit stages $\beta < 2^{\aleph_0}$ put $C_i^\beta = \bigcup \{C_i^\alpha : \alpha < \beta\}$, $i = 0, 1$. Finally put $C_0 = \bigcup \{C_0^\alpha : \alpha < 2^{\aleph_0}\}$ and $C_1 = (\omega)^2 \setminus C_0$. Clearly $Y_i^\alpha \in (X_\alpha)^2 \setminus C_{1-i}$ for $i = 0, 1$, so we have our counterexample.

The above construction made essential use of the Axiom of Choice. We shall shown in Section 5 that any proof of the existence of a counterexample must use the Axiom of Choice. Namely, there is a model of Zermelo–Fraenkel set theory without the Axiom of Choice in which Theorems 1.2 and 1.3 remain true even when the hypothesis "each C_i Borel" is dropped entirely. We obtain this result by dualizing a well known forcing construction of Mathias [24].

It is interesting to note that many well known combinatorial theorems can be deduced as corollaries of the main results of this paper. For instance, Theorem 2.2 is a slight generalization of the Dual Ramsey Theorem 1.2 in which partitions are replaced by A-partitions where A is a finite alphabet. Thus Theorem 2.2 may be viewed as an infinitary analog of a fairly difficult theorem of finite combinatorics due to Graham and Rothschild [11]. In Section 3 we deduce the Graham–Rothschild Theorem as a corollary of our infinitary result. We also deduce Ramsey's Theorem [32], the Halpern–Läuchli Theorem [14], and an "infinite-dimensional" generalization of the Halpern–Läuchli Theorem due to Laver [21]. (Unfortunately, Hindman's Theorem [16] does not seem to be easily deducible from the results in this paper. However, Hindman's Theorem as well as its topological generalization due to Milliken [26] are easily deduced from a theorem of Carlson [5] which is closely related to the results of Section 6. See Theorem 6.9 and Remark 6.10 below.)

We end this introduction with some historical remarks. In August 1981, subsequent to some conversations with Klaus Leeb [22], Simpson developed a series of conjectures which are stated as Theorems 2.2, 4.1, 5.7 and 5.8 below. Simpson's chief inspiration came from the theorems of Galvin–Prikry [10], Graham–Rothschild [11], and Paris–Harrington [29]. When Simpson tried to prove his conjectures, he succeeded only in establishing the special case $k = 3$ of Theorem 1.2. His proof of this special case used Hindman's Theorem [16]. Simpson also managed to reduce all of his conjectures to a certain infinitary Hales–Jewett [13] type conjecture which is stated below as Lemma 2.4. These reductions due to Simpson are presented in Sections 2, 3, 4 and 5 below. But Simpson's attempts to prove the key Lemma 2.4 met with no success. At that point Simpson communicated his conjectures to several people including Ron Graham and Leo Harrington.

Later, in July 1982, Simpson and Carlson met at the AMS Recursion Theory Institute which was held at Cornell University. In several conversations Simpson told Carlson of his conjectures and of his attempts to prove them. In particular Simpson described the key role of Lemma 2.4 and mentioned the relevance of Hindman's Theorem [16]. Carlson and Simpson discussed these matters further at an AMS meeting in Toronto in August 1982.

Shortly after the Toronto meeting, Carlson obtained a proof of Lemma 2.4 and indeed of the stronger Theorem 6.3. It is essentially that proof of Lemma 2.4 which we present below in Section 6. Subsequently, in October 1982, Carlson [4] devised a more difficult proof which yields a still stronger result, namely a common generalization of Lemma 2.4 and Hindman's Theorem [16] as well as the Hindman–Milliken Theorem [26]. (See Theorem 6.9 and Remark 6.10 below.) This more difficult proof of Carlson's was circulated in manuscript form by Prikry [31]. Carlson plans to publish it in a separate paper [5] which will also contain further results obtained by the same method.

2. Proof of the Dual Ramsey Theorem

The purpose of this section is to prove the Dual Ramsey Theorem 1.2. We find it convenient to prove a more general theorem in which partitions are replaced by A-partitions.

2.1. Definition. Let A be a fixed finite set of symbols which is disjoint from ω. We refer to A as a *finite alphabet*. An *A-partition* of ω is a collection of pairwise disjoint, nonempty subsets of $A \cup \omega$ called *blocks*, whose union is all of $A \cup \omega$, and such that each block contains at most one element of A. A *free block* is a block which is disjoint from A.

Let $(\omega)^\omega_A$ be the set of all A-partitions of ω having infinitely many free blocks. For $k \in \omega$ let $(\omega)^k_A$ be the set of all A-partitions of ω having exactly k free blocks. (Equivalently, $(\omega)^k_A$ is the set of all A-partitions of ω having exactly $|A| + k$ blocks. Here $|A|$ is the cardinality of A.) If X and Y are A-partitions of ω, we say that Y is *coarser than* X if each block of X is a subset of some block of Y. For $X \in (\omega)^\omega_A$ we write

$$(X)^\omega_A = \{Y \in (\omega)^\omega_A : Y \text{ is coarser than } X\}$$

and

$$(X)^k_A = \{Y \in (\omega)^k_A : Y \text{ is coarser than } X\}.$$

The main result of this section is the following:

2.2. Theorem. *Let A be a finite alphabet. If $(\omega)^k_A = C_0 \cup \cdots \cup C_{l-1}$, where each C_i is Borel, then there exists $X \in (\omega)^\omega_A$ such that $(X)^k_A \subseteq C_i$ for some i.*

The Dual Ramsey Theorem 1.2 is a special case of Theorem 2.2 obtained by taking $A = \varnothing$ = the empty set.

Before proving Theorem 2.2 we must develop some notation. We conform to the usual practice of identifying $n \in \omega$ with the set of all smaller natural numbers, i.e., $n = \{0, 1, \ldots, n-1\}$. For $X \in (\omega)^\omega_A$ we write $s < X$ to mean that s is a *segment of* X, i.e., $s = X|n|$ for some $n \in \omega$, where

$$X|n| = \{x \cap (A \cup n) : x \in X\} \setminus \{\varnothing\}.$$

In this case we write $lh(s) = n$ and $\#(s) = |\{x \in s : x \subseteq n\}|$.

By an *A-segment* we mean a segment of any $X \in (\omega)^\omega_A$. Thus an A-segment s is nothing more than an A-partition of $lh(s) \in \omega$, and $\#(s)$ is the number of free blocks of s. If s and t are A-segments, $s < t$ means that s is a segment of t, i.e., $s = t|n|$ for some $n < lh(t)$. Also $s \leqslant t$ means that $s < t$ or $s = t$. Also $s \leqslant s'$ means that $lh(s) = lh(s')$ and s is coarser than s', i.e., each block of s' is a subset of some block of s. Finally $s \leqslant X$ means that $s \leqslant X|lh(s)|$, or equivalently $s < Y$ for some $Y \in (X)^\omega_A$. If $s \leqslant X$ we write

$$(s, X)^\omega_A = \{Y \in (X)^\omega_A : s < Y\}$$

and

$$(s, X)^k_A = \{Y \in (X)^k_A : s < Y\}.$$

We shall now prove Theorem 2.2 for $k = 0$.

2.3. Lemma. *Let A be a finite alphabet. If $X \in (\omega)^\omega_A$ and $(X)^0_A = C_0 \cup \cdots \cup C_{l-1}$, where each C_i is Borel, then there exists $Y \in (X)^\omega_A$ such that $(Y)^0_A \subseteq C_i$ for some i.*

Proof. Note that $(X)^0_A$ is a compact Hausdorff space with basic open sets $(s, X)^0_A$, where $s \leqslant X$, $\#(s) = 0$. Since C_i is a Borel set, it has the property of