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LOGICAL ANALYSIS OF SOME THEOREMS OF COMBINATORICS AND TOPOLOGICAL DYNAMICS

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§1. Introduction.

Let $\mathbb{N}$ be the set of nonnegative integers. Given $X \subseteq \mathbb{N}$ let FS(X) be the set of all sums of finite nonempty subsets of X. Hindman's Theorem, HT, is the following statement.

$$(\mathrm{HT})\begin{cases}\underline{\text{If}}\quad \mathbb{N} = C_0 \cup \dots \cup C_\ell \quad \underline{\text{then}}\\ \underline{\text{there exists an infinite set}}\quad X \subseteq \mathbb{N}\\ \underline{\text{such that}}\quad FS(X) \subseteq C_i \quad \underline{\text{for some}}\quad i \le \ell.\end{cases}$$

It is well known that all existing proofs of HT are nonconstructive. One of the goals of this paper is to delimit the degree of nonconstructivity which is inherent in Hindman's Theorem. We also discuss some related theorems from combinatorics (Carlson-Simpson) and topological dynamics (Auslander-Ellis).

Our results concerning Hindman's Theorem are of two kinds: axiomatic and recursion-theoretic. The axiomatic results provide partial answers to the following question: Which set existence axioms are sufficient and/or necessary to prove HT? The recursion-theoretic results respond to a somewhat different question. Namely, what can one say about the recursion-theoretic complexity of the homogeneous set X relative to that of the given coloring $C_0,\dots,C_\ell$?

Our recursion-theoretic work has its precedent in Jockusch's recursion-theoretic analysis of Ramsey's Theorem [17]. Regrettably, our results on Hindman's Theorem are not so complete as those of Jockusch on Ramsey's Theorem. By adapting a device of Jockusch, we prove in §2 the

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following negative result. For all $W \subseteq \mathbb{N}$ there exists a coloring $\mathbb{N} = C_0 \cup C_1$ which is recursive in W, such that for all infinite sets $X \subseteq \mathbb{N}$ and $i \in \{0,1\}$, $FS(X) \subseteq C_i$ implies that X is not recursive in $W^{(1)}$. (Here $W^{(1)}$ denotes the Turing jump of W.) We also prove a similar result with the conclusion "$W^{(1)}$ is recursive in X" in place of "X is not recursive in $W^{(1)}$". In §4 we obtain the following positive result. For all $W \subseteq \mathbb{N}$, if a given coloring $\mathbb{N} = C_0 \cup \ldots \cup C_\ell$ is recursive in W, then there exists an infinite set $X \subseteq \mathbb{N}$ such that $FS(X) \subseteq C_i$ for some $i \leq \ell$, and X is recursive in $W^{(\omega+1)}$. (Here $W^{(\alpha)}$ denotes the αth Turing jump of W.) Thus we have lower and upper bounds $W^{(1)}$ and $W^{(\omega+1)}$ for the recursion-theoretic complexity of X. It would be desirable to narrow or close the gap between these two bounds.

There is a rather extensive literature on Hindman's Theorem. See for instance the papers by Blass [2] and Hindman [16] in this volume. There are four known proofs of Hindman's Theorem: (1) the original combinatorial proof due to Hindman [15]; (2) the simplified combinatorial proof due to Baumgartner [1]; (3) the dynamical proof due to Furstenberg and Weiss [10], [9]; and (4) the ultrafilter proof due to Glazer [12]. A convenient reference for proofs (2), (3) and (4) is the book by Graham, Rothschild and Spencer [13].

Our results in §4 are based on a somewhat delicate analysis of Hindman's original proof. This analysis yields the above-mentioned, recursion-theoretic upper bound. In axiomatic terms, the same analysis shows that Hindman's Theorem is provable in a certain formal system ACA_0^+. Namely ACA_0^+ is the subsystem of second order arithmetic whose principal axiom asserts that arithmetical comprehension can be iterated along the natural numbers. (For information on subsystems of second order arithmetic, see [25], [5], [24], [8], [4].)

In §3 we present a somewhat similar analysis of Baumgartner's proof. This analysis yields no recursion-theoretic information beyond what is provided automatically by the Kleene Basis Theorem. However, the analysis does lead to an interesting axiomatic conclusion. Namely, Baumgartner's proof or something like it can be pushed through in the formal system Π_2^1-TI_0 (described in §3). This conclusion is interesting because it applies not only to Hindman's Theorem but also to other results which are proved by methods similar to that of Baumgartner. For instance, Theorem 6.3 of Carlson-Simpson [6] is provable in Π_2^1-TI_0. We do not know whether Theorem 6.3 of Carlson-Simpson [6] is provable in any weaker system, e.g. RCA_0 or ACA_0^+ or Δ_2^1-TI_0.

Furstenberg and Weiss [10], [9] (see also [13]) have made the following very interesting observation: Hindman's Theorem can be deduced rather easily from a theorem of topological dynamics due to Auslander and Ellis. The *Auslander-Ellis Theorem*, AET, reads as follows.

(AET)
Let X *be a compact metric space*
and let $T: X \to X$ *be continuous*.
Regard $(X, \langle T^n \rangle_{n \in \mathbb{N}})$ *as a dynamical*
system. *Given* $x \in X$, *there exists*
$y \in X$ *such that* y *is uniformly*
recurrent and proximal to x.

For an explanation of the notions of uniform recurrence and proximality, see e.g. [9] or [13] or §5 below.

The purpose of §5 is to present an axiomatic analysis of AET. The classical proof of AET is extremely nonconstructive, relying as it does on Zorn's Lemma applied to the partial ordering by inclusion of the closed subsets of the nonmetrizable Tychonoff product space X^X. (See the discussion of the "enveloping semigroup" on page 159 of [9] or page 143 of [13].) It is not at all obvious that this classical proof or anything like it can be carried out within full second order arithmetic. In §5 we present an apparently new proof of AET in which Hindman's Theorem is used as a lemma. We show that all parts of the new proof, except possibly the applications of Hindman's Theorem, can be pushed through in ACA_0. Combining this with a result from §4, we conclude: AET is provable in ACA_0^+. Thus our proof of AET is much closer to being constructive than is the classical proof.

§2. *Strong recursive counterexample to Hindman's Theorem*.

Given $X, W \subseteq \mathbb{N}$ we say that X *is recursive in* W if the characteristic function of X is computable by a Turing machine using an oracle for the characteristic function of W. We use $W^{(1)}$ to denote the *Turing jump* of W. In particular $\emptyset^{(1)}$ is the Turing jump of the empty set, i.e. the complete recursively enumerable subset of $\mathbb{N}$. Thus

$\emptyset^{(1)}$ has the same degree of unsolvability as the Halting Problem. The recursion-theoretic notions which we use are explained in Rogers [21].

The purpose of this section is to prove the following theorems.

2.1. *Theorem. There exists a recursive coloring* $\mathbb{N} = C_0 \cup C_1$ *such that for all infinite* $X \subseteq \mathbb{N}$, *if* $FS(X) \subseteq C_i$ *for some* $i \in \{0,1\}$, *then* X *is not recursive in* $\emptyset^{(1)}$.

2.2. *Theorem. There exists a recursive coloring* $\mathbb{N} = C_0 \cup C_1$ *such that, for all infinite* $X \subseteq \mathbb{N}$, *if* $FS(X) \subseteq C_i$ *for some* $i \in \{0,1\}$, *then* $\emptyset^{(1)}$ *is recursive in* X.

Proof of Theorem 2.1. We imitate the proof of Theorem 3.1 of Jockusch [17].

For $A \subseteq \mathbb{N}$ let $c_A\colon \mathbb{N} \to \{0,1\}$ be the characteristic function of A. By Theorem 2 of Shoenfield [22] there exists a recursive function $f\colon \mathbb{N}^3 \to \mathbb{N}$ with the following property. For all $A \subseteq \mathbb{N}$, A is recursive in $\emptyset^{(1)}$ if and only if, for some j, $c_A(u) = \lim_s f(j,u,s)$ for all $u \in \mathbb{N}$. Let us write $A = A_j$ in this case.

If A_j is defined and has at least $2j+2$ elements, let D_j consist of the smallest $2j+2$ elements of A_j. Otherwise let D_j be undefined. We shall now define a finite set D_j^s to approximate D_j at stage s. If there are at least $2j+2$ numbers u such that $u \leq s$ and $f(j,u,s) = 1$, let D_j^s consist of the smallest $2j+2$ such numbers. Otherwise let D_j^s be undefined.

Given $n \geq 1$ let us write $\lambda(n) = n_1$ and $\mu(n) = n_k$ where $n = 2^{n_1} + \ldots + 2^{n_k}$, $n_1 < \ldots < n_k$. Note that $\lambda(m+n) = \lambda(m)$ and $\mu(m+n) = \mu(n)$ provided $\mu(m) < \lambda(n)$.

The recursive coloring $\mathbb{N} = C_0 \cup C_1$ will be constructed in stages. At stage s of the construction we shall place each of the finitely many numbers n with $\mu(n) = s$ into exactly one of the color classes C_0 and C_1.

Stage s. By induction on $j \leq s$, let u_j^s and v_j^s be two effectively chosen numbers which are different from each other and from all u_i^s and v_i^s, $i < j$, and which belong to D_j^s if D_j^s is defined. This can be done since $|D_j^s| = 2j + 2$ if D_j^s is defined. Now for all n such that $\mu(n) = s$, put $n \in C_0$ if $\lambda(n) = u_j^s$ for some $j \leq s$, otherwise $n \in C_1$.

This completes the construction. Clearly C_0 and C_1 are recursive.

Let X be an infinite set such that $FS(X) \subseteq C_0$ or $FS(X) \subseteq C_1$. We claim that X is not recursive in $\emptyset^{(1)}$. To see this we first let Y be an infinite set such that Y is recursive in X, $FS(Y) \subseteq FS(X)$, and $\mu(m) < \lambda(n)$ for all $m \in Y$, $n \in Y$, $m < n$. (See Lemma 4.1 below.) Put $Z = \{\lambda(n): n \in Y\}$. Suppose that X is recursive in $\emptyset^{(1)}$. Then Z is recursive in $\emptyset^{(1)}$ so let j be such that $Z = A_j$. Since Z is infinite, D_j is defined and $D_j \subseteq Z$. Choose $n \in Y$ so large that $\max(D_j) < \lambda(n)$ and $D_j^s = D_j$ where $s = \mu(n)$. Then u_j^s and v_j^s are distinct elements of $D_j^s = D_j \subseteq Z$. Let $m_0, m_1 \in Y$ be such that $\lambda(m_0) = u_j^s$ and $\lambda(m_1) = v_j^s$. Then $\max(\mu(m_0), \mu(m_1)) < \lambda(n)$, hence $m_0 + n, m_1 + n \in FS(Y) \subseteq FS(X)$ and $m_0 + n \in C_0$, $m_1 + n \in C_1$. This contradiction completes the proof.

Proof of Theorem 2.2. We view each $n \in \mathbb{N}$ as a code for the finite set $\{n_1, \ldots, n_k\}$, where $n = 2^{n_1} + \ldots + 2^{n_k}$ and $n_1 < \ldots < n_k$.