

Almost everywhere domination and superhighness

Stephen G. Simpson
Pennsylvania State University
<http://www.math.psu.edu/simpson/>
simpson@math.psu.edu

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Abstract

Let ω denote the set of natural numbers. For functions $f, g : \omega \rightarrow \omega$, we say that f is dominated by g if $f(n) < g(n)$ for all but finitely many $n \in \omega$. We consider the standard “fair coin” probability measure on the space 2^ω of infinite sequences of 0’s and 1’s. A Turing oracle B is said to be almost everywhere dominating if, for measure one many $X \in 2^\omega$, each function which is Turing computable from X is dominated by some function which is Turing computable from B . Dobrinen and Simpson have shown that the almost everywhere domination property and some of its variant properties are closely related to the reverse mathematics of measure theory. In this paper we exposit some recent results of Kjos-Hanssen, Kjos-Hanssen/Miller/Solomon, and others concerning LR -reducibility and almost everywhere domination. We also prove the following new result: If B is almost everywhere dominating, then B is superhigh, i.e., $0''$ is truth-table computable from B' , the Turing jump of B .

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1 Introduction

The concept of almost everywhere domination was originally introduced by Dobrinen and Simpson [7] with applications to the reverse mathematics of measure theory [26, Section X.1]. Subsequent work by Binns, Cholak, Greenberg, Kjos-Hanssen, Lerman, Miller, and Solomon [2, 5, 13, 14] has greatly illuminated this concept and established its close relationship to the decisive results on K -triviality and low-for-randomness which are due to Downey, Hirschfeldt, Kučera, Nies, Stephan, and Terwijn [16, 9, 21, 11]. The purpose of this paper is to update the Dobrinen/Simpson account of almost everywhere domination by expositing this subsequent research. We provide introductory accounts of Martin-Löf randomness, LR -reducibility, and prefix-free Kolmogorov complexity as they relate to almost everywhere domination. We also prove a new result: If B is almost everywhere dominating, then B is superhigh.

The reader who is familiar with the basic concepts and results of recursion theory will find that our exposition in this paper is self-contained, except for some peripheral remarks. Throughout this paper we give full proofs and strive for simplicity and clarity.

2 Notation

We use standard recursion-theoretic notation and terminology from Rogers [25]. We write *r.e.* as an abbreviation for “recursively enumerable”. If C is a Turing oracle, we write *C*-recursive for “recursive relative to C ”, *C*-r.e. for “recursively enumerable relative to C ”, etc. We write $\omega = \{0, 1, \dots, n, \dots\}$ to denote the set of natural numbers. We often identify Turing oracles with subsets of ω . If E is an expression denoting a natural number which may or may not be defined, we write $E \downarrow$ to mean that E is defined, and $E \uparrow$ to mean that E is undefined. If E_1 and E_2 are two such expressions, we write $E_1 \simeq E_2$ to mean that E_1 and E_2 are both undefined or both defined with the same value. If C is a Turing oracle, we write

$$C' = \{e \in \omega \mid \varphi_e^{(1),C}(0) \downarrow\} = \text{the Turing jump of } C.$$

In particular $O' = \{e \in \omega \mid \varphi_e^{(1)}(0) \downarrow\}$ is a Turing oracle for the Halting Problem. For $A, B \subseteq \omega$ we write

$$A \oplus B = \{2n \mid n \in A\} \cup \{2n + 1 \mid n \in B\},$$

the *Turing join* of A and B . We write \leq_T to denote *Turing reducibility*. Thus $A \leq_T B$ means that A is Turing computable from B . For $A \subseteq \omega$ we write $\overline{A} = \omega \setminus A$, the complement of A . We sometimes identify $A \subseteq \omega$ with its characteristic function $\chi_A : \omega \rightarrow \{0, 1\}$ given by $\chi_A(n) = 1$ if $n \in A$, 0 if $n \notin A$.

We write 2^ω to denote the *Cantor space*, i.e., the set of total functions $X : \omega \rightarrow \{0, 1\}$. We write $2^{<\omega}$ to denote the set of *strings*, i.e., finite sequences of 0's and 1's. For $\sigma \in 2^{<\omega}$ we write $\sigma = \langle i_0, \dots, i_{n-1} \rangle$ where $n = |\sigma|$ is the length of σ . The *empty string*, denoted $\langle \rangle$, is the unique string of length 0. Given $X \in 2^\omega$ and $n \in \omega$, we have a string $X \upharpoonright n = \langle X(0), \dots, X(n-1) \rangle$ of length n . For $\sigma \in 2^{<\omega}$ and $X \in 2^\omega$, we write $\sigma \subset X$ to mean that σ is a *prefix* of X , i.e., $\sigma = X \upharpoonright |\sigma|$. For $\sigma, \tau \in 2^{<\omega}$, we write $\sigma \subset \tau$ to mean that σ is a *prefix* of τ , i.e., a proper initial segment of τ . We write $\sigma \hat{\ } \tau$ = the concatenation, σ followed by τ . Thus $|\sigma \hat{\ } \tau| = |\sigma| + |\tau|$. Moreover, $\sigma \subset \tau$ if and only if $\sigma \hat{\ } \rho = \tau$ for some $\rho \neq \langle \rangle$. For $\sigma \in 2^{<\omega}$ and $X \in 2^\omega$ we write $\sigma \hat{\ } X$ = the concatenation, σ followed by X . Thus $\sigma \subset X$ if and only if $\sigma \hat{\ } Y = X$ for some Y .

Given $\sigma \in 2^{<\omega}$, we write $N_\sigma = \{X \in 2^\omega \mid \sigma \subset X\}$. Thus N_σ is a basic open neighborhood in the Cantor space. The *fair-coin probability measure* μ on 2^ω is defined by $\mu(N_\sigma) = 1/2^{|\sigma|}$. In particular $\mu(2^\omega) = 1$. A set $S \subseteq 2^{<\omega}$ is said to be *prefix-free* if there are no $\sigma, \tau \in S$ such that $\sigma \subset \tau$. Note that if S is prefix-free then $\mu(\bigcup_{\sigma \in S} N_\sigma) = \sum_{\sigma \in S} 1/2^{|\sigma|}$.

We make extensive use of the relativized arithmetical hierarchy of subsets of 2^ω . See Rogers [25, Section 15.1]. If C is a Turing oracle, then by definition $U \subseteq 2^\omega$ is $\Sigma_1^{0,C}$ if and only if $U = \bigcup_{\sigma \in S} N_\sigma$ for some C -r.e. set $S \subseteq 2^{<\omega}$. A well known fact is that for any such U we can find such an S which is prefix-free. By definition, $P \subseteq 2^\omega$ is $\Pi_1^{0,C}$ if and only if its complement $\overline{P} = 2^\omega \setminus P$ is $\Sigma_1^{0,C}$. Note that $\mu(P) = 1 - \mu(\overline{P})$. Because 2^ω is compact, a $\Sigma_1^{0,C}$ set $U \subseteq 2^\omega$ is $\Pi_1^{0,C}$ if and only if $U = \bigcup_{\sigma \in F} N_\sigma$ for some finite $F \subseteq 2^{<\omega}$. This is the same as

saying that U is *clopen*, i.e., closed and open, in 2^ω . Again, we may take F to be prefix-free.

3 Randomness

Our work in this paper is based on a robust concept of randomness relative to a Turing oracle. The original, unrelativized concept is due to Martin-Löf [17] and has been studied by Kučera [15] and many others.

Definition 3.1 (Martin-Löf 1966). Let C be a Turing oracle. We say that $X \in 2^\omega$ is C -random if $X \notin \bigcap_n U_n^C$ for all uniformly $\Sigma_1^{0,C}$ sequences of sets $U_n^C \subseteq 2^\omega$, $n = 0, 1, 2, \dots$ such that $\mu(U_n^C) \leq 1/2^n$ for all n . Such a sequence is called a *Martin-Löf test* for C -randomness.

Theorem 3.2 (Martin-Löf 1966, Kučera 1985). *We can construct a universal Martin-Löf test. In other words, we can find uniformly $\Sigma_1^{0,C}$ sets U_n^C , $n = 0, 1, 2, \dots$ such that $\mu(U_n^C) \leq 1/2^n$ and, for all $X \in 2^\omega$ and all Turing oracles C , X is C -random if and only if $X \notin \bigcap_n U_n^C$.*

Proof. Let V_i^C , $i = 0, 1, 2, \dots$ be a standard, uniform enumeration of all $\Sigma_1^{0,C}$ subsets of 2^ω . For $s = 0, 1, \dots$ let $V_{i,s}^C$ be the set of $X \in 2^\omega$ which get into V_i^C by means of a computation in $\leq s$ steps using only oracle information from $C \upharpoonright s$. Thus $V_{i,0}^C \subseteq V_{i,1}^C \subseteq \dots \subseteq V_{i,s}^C \subseteq \dots$, $s = 0, 1, \dots$ is a uniformly C -recursive sequence of clopen sets, and $V_i^C = \bigcup_s V_{i,s}^C$. For rational numbers r let

$$V_i^C[r] = \bigcup_{\mu(V_{i,s}^C) \leq r} V_{i,s}^C.$$

Intuitively, $V_i^C[r]$ is just V_i^C enumerated so long as its measure is $\leq r$. Note that $V_i^C[r]$ is uniformly $\Sigma_1^{0,C}$, and $\mu(V_i^C[r]) \leq r$, and $V_i^C[r] = V_i^C$ if and only if $\mu(V_i^C) \leq r$. Now for all $e, n \in \omega$ let $\tilde{U}_{e,n}^C = V_i^C[1/2^n]$ if $\varphi_e^{(1)}(n) \simeq i$, and \emptyset if $\varphi_e^{(1)}(n) \uparrow$. Again $\tilde{U}_{e,n}^C$ is uniformly $\Sigma_1^{0,C}$. Moreover, for each e , the sequence $\tilde{U}_{e,n}^C$, $n = 0, 1, \dots$ is a Martin-Löf test, and all Martin-Löf tests occur in this way. Finally let $U_n^C = \bigcup_e \tilde{U}_{e,e+n+1}^C$. Then U_n^C is uniformly $\Sigma_1^{0,C}$, and $\mu(U_n^C) \leq \sum_e 1/2^{e+n+1} = 1/2^n$ for all n , so U_n^C , $n = 0, 1, \dots$ is a Martin-Löf test. Moreover, for an arbitrary Martin-Löf test $\tilde{U}_{e,n}^C$, $n = 0, 1, \dots$, we have $\bigcap_n \tilde{U}_{e,n}^C \subseteq \bigcap_n U_n^C$. Thus U_n^C , $n = 0, 1, \dots$ is a universal Martin-Löf test. \square

Corollary 3.3 (Kučera 1985). *For any Turing oracle C , the set*

$$R^C = \{X \in 2^\omega \mid X \text{ is } C\text{-random}\}$$

is uniformly $\Sigma_2^{0,C}$ and of measure 1.

Proof. By Theorem 3.2 let U_n^C , $n = 0, 1, \dots$ be a universal Martin-Löf test for C -randomness. Then $R^C = 2^\omega \setminus \bigcap_n U_n^C$. Moreover U_n^C is uniformly $\Sigma_1^{0,C}$, hence

$\bigcap_n U_n^C$ is uniformly $\Pi_2^{0,C}$, hence R^C is uniformly $\Sigma_2^{0,C}$. Also $\mu(U_n^C) \leq 1/2^n$ for all n , hence $\mu(\bigcap_n U_n^C) = 0$, hence $\mu(R^C) = 1$. \square

We now present van Lambalgen's Theorem, from [30].

Lemma 3.4. *Assume that $A \oplus B$ is random. Then A is B -random, and B is A -random. In particular, A and B are random.*

Proof. Suppose for instance that B is not A -random. Then $B \in \bigcap_n V_n^A$ where V_n^A is uniformly $\Sigma_1^{0,A}$ of measure $\leq 1/2^n$. Let $W_n = \{X \oplus Y \mid X \in 2^\omega, Y \in V_n^X[1/2^n]\}$ and note that W_n is uniformly Σ_1^0 of measure $\leq 1/2^n$. We have $A \oplus B \in \bigcap_n W_n$, contradicting the assumption that $A \oplus B$ is random. \square

Lemma 3.5 (Solovay's Lemma). *Assume A is random. Let $U_n \subseteq 2^\omega$, $n = 0, 1, \dots$ be uniformly Σ_1^0 such that $\sum_{n=0}^\infty \mu(U_n) < \infty$. Then $\exists^{<\infty} n (A \in U_n)$, i.e., $\{n \mid A \in U_n\}$ is finite.*

Proof. For each k let $W_k = \{X \mid \exists^{\geq k} n (X \in U_n)\}$. Let c be such that $\sum_{n=0}^\infty \mu(U_n) \leq c$. We claim that $\mu(W_k) \leq c/k$. To see this, let $W_k^N = \{X \mid (\exists^{\geq k} n \leq N) (X \in U_n)\}$ and note that $W_k = \bigcup_N W_k^N$. For all N we have

$$\begin{aligned} c &\geq \sum_{n=0}^\infty \mu(U_n) \geq \sum_{n=0}^N \mu(U_n) = \sum_{n=0}^N \int_{2^\omega} U_n(X) dX \\ &= \int_{2^\omega} \sum_{n=0}^N U_n(X) dX \geq \int_{2^\omega} k W_k^N(X) dX = k \mu(W_k^N) \end{aligned}$$

so $\mu(W_k^N) \leq c/k$ for all N . It follows that $\mu(W_k) \leq c/k$, and our claim is proved. Since A is random and W_k is uniformly Σ_1^0 , we have $A \notin \bigcap_k W_k$, hence $A \notin W_k$ for some k , hence $\exists^{<k} n (A \in U_n)$. This proves the lemma. \square

Theorem 3.6 (van Lambalgen's Theorem). *$A \oplus B$ is random if and only if A is random and B is A -random.*

Proof. The "only if" direction follows from Lemma 3.4. For the "if" direction, assume that $A \oplus B$ is not random. We have $A \oplus B \in \bigcap_n W_n$ where W_n is uniformly Σ_1^0 with $\mu(W_n) \leq 1/2^n$. By passing to a subsequence, we may assume that $\mu(W_n) \leq 1/2^{2n}$. Let $U_n = \{X \mid \mu(\{Y \mid X \oplus Y \in W_n\}) > 1/2^n\}$ and note that U_n is uniformly Σ_1^0 . Moreover $\mu(U_n) \leq 1/2^n$ for all n , because otherwise we would have

$$\mu(W_n) > \mu(U_n) \cdot \frac{1}{2^n} > \frac{1}{2^n} \cdot \frac{1}{2^n} = \frac{1}{2^{2n}},$$

a contradiction. Since A is random, it follows by Lemma 3.5 that $\{n \mid A \in U_n\}$ is finite. Thus for all but finitely many n we have $A \notin U_n$, i.e., $\mu(\{Y \mid A \oplus Y \in W_n\}) \leq 1/2^n$. Let $V_n^A = \{Y \mid A \oplus Y \in W_n\}$. Then $\mu(V_n^A) \leq 1/2^n$ for all but finitely many n , and V_n^A is uniformly $\Sigma_1^{0,A}$. Moreover $B \in \bigcap_n V_n^A$, contradicting the assumption that B is A -random. \square

Next we present the Kučera/Gács Theorem, from Kučera [15].

Lemma 3.7. *Let $P \subseteq 2^\omega$ be Π_1^0 of positive measure. Then for all $X \in 2^\omega$ there exists $Y \in P$ such that $Y \leq_T X \oplus 0'$ and $X \leq_T Y$.*

Proof. Claim: If $\mu(P \cap N_\sigma) \geq 1/2^k$ where $k \geq 1$, then there are at least two strings $\tau \supset \sigma$ such that $|\tau| = 2k$ and $\mu(P \cap N_\tau) \geq 1/2^{4k}$.

To prove the claim, note first that $\mu(N_\sigma) = 1/2^{|\sigma|} \geq 1/2^k$, hence $|\sigma| \leq k < 2k$ since $k \geq 1$. If the conclusion of the claim were false, we would have

$$\begin{aligned} \mu(P \cap N_\sigma) &= \sum_{\tau \supset \sigma, |\tau|=2k} \mu(P \cap N_\tau) \\ &\leq \frac{1}{2^{2k}} + \left(2^{2k-|\sigma|} - 1\right) \frac{1}{2^{4k}} < \frac{1}{2^{2k}} + 2^{2k} \frac{1}{2^{4k}} \leq \frac{1}{2^k} \end{aligned}$$

contradicting the hypothesis of the claim.

To prove Lemma 3.7, assume that $P \subseteq 2^\omega$ is Π_1^0 with $\mu(P) > 0$, say $\mu(P) \geq 1/2^k$ where $k \geq 1$. Note that $N_\langle \rangle = 2^\omega$, hence $\mu(P \cap N_\langle \rangle) = \mu(P) \geq 1/2^k$. Applying our claim repeatedly, define $f : 2^{<\omega} \rightarrow 2^{<\omega}$ as follows. Let $f(\langle \rangle) = \langle \rangle$. Suppose inductively that $f(\rho)$ has already been defined. Let $k_i = 4^i k$ where $i = |\rho|$. Let $f(\rho^\wedge(0))$ (respectively $f(\rho^\wedge(1))$) be the lexicographically leftmost (respectively rightmost) $\tau \supset f(\rho)$ such that $|\tau| = 2k_i$ and $\mu(P \cap N_\tau) \geq 1/2^{4k_i}$. Our claim implies that $f(\rho^\wedge(0))$ and $f(\rho^\wedge(1))$ exist and are incompatible. It is straightforward to check that $f \leq_T 0'$.

Given $X \in 2^\omega$, let $Y = f(X) = \bigcup_i f(X \upharpoonright i)$. Clearly $Y \in P$ and $Y \leq_T X \oplus 0'$. We claim that $X \leq_T Y$. To prove this, we describe how to compute X using an oracle for Y . Let $P = \bigcap_s P_s$ where $P_0 \supseteq P_1 \supseteq \dots \supseteq P_s \supseteq \dots$ is a recursive sequence of clopen sets. Suppose we have already computed $\rho = X \upharpoonright i$. We also have $f(\rho) = Y \upharpoonright 2k_{i-1}$ if $i > 0$, or $\langle \rangle$ if $i = 0$. Our construction implies that $Y \upharpoonright 2k_i$ is either the leftmost or the rightmost $\tau \supset f(\rho)$ of length $2k_i$ such that $\mu(P \cap N_\tau) \geq 1/2^{4k_i}$. Therefore, for all sufficiently large stages s , we will have $\mu(P_s \cap N_\tau) < 1/2^{4k_i}$ for all $\tau \supset f(\rho)$ of length $2k_i$ lying to the left (respectively right) of $Y \upharpoonright 2k_i$. When such a stage s is reached, then at that point we see that $X(i) = 0$ (respectively $X(i) = 1$). This completes the proof. \square

Theorem 3.8 (Kučera/Gács Theorem). *For any $X \geq_T 0'$ we can find a random Y such that $Y \equiv_T X$.*

Proof. By Corollary 3.3 let $P \subseteq 2^\omega$ be Π_1^0 of positive measure such that $\forall Y (Y \in P \Rightarrow Y \text{ is random})$. Applying Lemma 3.7 to any $X \geq_T 0'$, we obtain $Y \in P$ such that $X \equiv_T Y$. \square

Corollary 3.9. *Assume that A is random and $A \leq_T B$ and B is C -random and $C \geq_T 0'$. Then A is C -random.*

Proof. Since $C \geq_T 0'$, we may assume by the Kučera/Gács Theorem that C is random. Since B is C -random and C is random, it follows by van Lambalgen's Theorem that $B \oplus C$ is random, hence C is B -random. Since $A \leq_T B$, it follows that C is A -random. Now, since A is random, it follows that $A \oplus C$ is random, hence A is C -random. \square

Remarkably, the previous corollary holds even without the assumption $C \geq_T 0'$. We mention without proof the following theorem of Miller/Yu [19].

Theorem 3.10 (Miller/Yu 2004). *If A is random and $A \leq_T B$ where B is C -random, then A is C -random.*

4 LR -reducibility

In this section we study the following reducibility notion, which was originally introduced by Nies [21, Section 8].

Definition 4.1 (Nies 2002). Let A and B be Turing oracles. We say that A is LR -reducible to B , abbreviated $A \leq_{LR} B$, if

$$\forall X (X \text{ is } B\text{-random} \Rightarrow X \text{ is } A\text{-random}).$$

Remark 4.2. Obviously the binary relation \leq_{LR} is transitive and reflexive. Note also that, as a reducibility relation, \leq_{LR} is at least as coarse as Turing reducibility. In other words, $A \leq_T B$ implies $A \leq_{LR} B$. In Section 6 we shall construct an $A \leq_{LR} 0$ such that $A \not\leq_T 0$, i.e., A is not recursive. Thus \leq_{LR} does not coincide with \leq_T .

Remark 4.3. Evidently the reducibility relation \leq_{LR} is closely related to the notion of low-for-randomness, which was first introduced by Zambella [31] and has been studied extensively by Kučera/Terwijn [16], Terwijn/Zambella [29], Downey/Hirschfeldt/Nies/Stephan [9], Hirschfeldt/Nies/Stephan [11], and Nies [21]. By definition, A is *low-for-random* if and only if $A \leq_{LR} 0$. Relativizing to B , we see that A is low-for-random relative to B if and only if $A \oplus B \leq_{LR} B$.

However, caution is required, because in general $A \leq_{LR} B$ is not equivalent to A being low-for-random relative to B . In Section 6 we shall construct a Turing oracle C such that $0' \leq_{LR} C$ yet $0'$ is not low-for-random relative to C . We shall also see that the binary relation “ A is low-for-random relative to B ” is not transitive. Indeed, we shall construct Turing oracles B and C such that $B \leq_T 0' \leq_{LR} B \leq_T C$, hence $0'$ is low-for-random relative to B and B is low-for-random relative to C , yet $0'$ is not low-for-random relative to C . See Theorem 6.10.

Remark 4.4. Another way to distinguish \leq_{LR} from relative low-for-randomness is as follows. It can be shown (see Lemma 7.4 and Remark 10.12) that if A is low-for-random relative to B then $A \leq_T B'$, hence for each B there are only countably many such A . But Miller and Yu [19, 18] have constructed a B such that $\{A \mid A \leq_{LR} B\}$ is uncountable. Recently Barmpalias/Lewis/Soskova [1] have shown that this holds for any B which is generalized superhigh.

Remark 4.5. On the other hand, consider the equivalence relation \equiv_{LR} defined by letting $A \equiv_{LR} B$ if and only if $A \leq_{LR} B$ and $B \leq_{LR} A$. It can be shown (see Remark 10.12) that if $A \equiv_{LR} B$ then A is low-for-random relative to B and B is low-for-random relative to A . It follows that each \equiv_{LR} -equivalence class is countable.

We now prove the following theorem due to Kjos-Hanssen [13].

Theorem 4.6 (Kjos-Hanssen 2005). *The following statements are pairwise equivalent.*

1. $A \leq_{LR} B$.
2. Each $\Pi_1^{0,A}$ set of positive measure includes a $\Pi_1^{0,B}$ set of positive measure.
3. There exists a $\Pi_1^{0,A}$ set P such that $\forall X (X \in P \Rightarrow X \text{ is } A\text{-random})$ and P includes a $\Pi_1^{0,B}$ set of positive measure.
4. There exists a $\Pi_1^{0,B}$ set Q of positive measure such that $\forall Y (Y \in Q \Rightarrow Y \text{ is } A\text{-random})$.

Proof. In order to prove Theorem 4.6, we first prove several lemmas.

Lemma 4.7. *Fix a Turing oracle B , and let $P \subseteq 2^\omega$ be $\Pi_1^{0,B}$. The following statements are pairwise equivalent.*

- (a) P is of positive measure.
- (b) $\exists Q (Q \subseteq P \text{ and } Q \text{ is nonempty } \Pi_1^{0,B} \text{ and } \forall X (X \in Q \Rightarrow X \text{ is } B\text{-random}))$.
- (c) $\exists X (X \in P \text{ and } X \text{ is } B\text{-random})$.

Proof. By Corollary 3.3, $\{X \in 2^\omega \mid X \text{ is } B\text{-random}\}$ is $\Sigma_2^{0,B}$ and of measure 1. From this it follows easily that (a) \Rightarrow (b). Trivially (b) \Rightarrow (c). In order to prove (c) \Rightarrow (a), assume $\mu(P) = 0$. We have $P = \bigcap_s P_s$ where $P_0 \supseteq P_1 \supseteq \dots \supseteq P_s \supseteq \dots$ is a B -recursive sequence of clopen sets. Hence $\lim_s \mu(P_s) = \mu(P) = 0$. Let $U_k = P_{f(k)}$ where $f(k) =$ least s such that $\mu(P_s) \leq 1/2^k$. Then $U_k, k = 1, 2, \dots$ is a Martin-Löf test for B -randomness, and $P = \bigcap_k U_k$. Hence no $X \in P$ is B -random, Q.E.D. \square

Lemma 4.8. *Assume that Q is $\Pi_1^{0,B}$ such that $\forall X (X \in Q \Rightarrow X \text{ is } B\text{-random})$. Then $\forall \sigma (Q \cap N_\sigma \neq \emptyset \Rightarrow Q \cap N_\sigma \text{ is of positive measure})$.*

Proof. This follows easily from Lemma 4.7. \square

The proof of Theorem 4.6 is based on the following idea, which goes back to Kučera [15].

Definition 4.9 (Kučera 1985). Let $U, V \subseteq 2^\omega$ be $\Sigma_1^{0,A}$. We define a product operation. Let $U = \bigcup_{\sigma \in S} N_\sigma$ and $V = \bigcup_{\tau \in T} N_\tau$ where $S, T \subseteq 2^{<\omega}$ are A -r.e. and prefix-free. We define the product

$$UV = \bigcup_{\sigma \in S, \tau \in T} N_{\sigma \hat{\ } \tau}.$$

Note that $\{\sigma \hat{\ } \tau \mid \sigma \in S, \tau \in T\}$ is again A -r.e. and prefix-free. Note also that:

- (a) UV is $\Sigma_1^{0,A}$.
- (b) Given indices of U and V qua $\Sigma_1^{0,A}$ sets, we can compute an index of UV qua $\Sigma_1^{0,A}$ set.
- (c) $UV \subseteq U$.
- (d) $\mu(UV) = \mu(U)\mu(V)$.
- (e) The product is associative, i.e., $(UV)W = U(VW)$.

Definition 4.10 (Kučera 1985). Dually, let $P, Q \subseteq 2^\omega$ be $\Pi_1^{0,A}$. We define the product $PQ = \overline{\overline{P}Q}$, where $\overline{P} = 2^\omega \setminus P$. Note that:

- (a) PQ is $\Pi_1^{0,A}$.
- (b) Given indices of P and Q qua $\Pi_1^{0,A}$ sets, we can compute an index of PQ qua $\Pi_1^{0,A}$ set.
- (c) $PQ \supseteq P$.
- (d) $\mu(PQ) = 1 - (1 - \mu(P))(1 - \mu(Q))$.
- (e) The product is associative, i.e., $(PQ)R = P(QR)$.

We now begin the proof of Theorem 4.6. Let us say that $P \subseteq 2^\omega$ is *fat* if it includes a $\Pi_1^{0,B}$ set of positive measure. The implication $1 \Rightarrow 2$ of Theorem 4.6 may be rephrased as follows:

If $A \leq_{LR} B$, then every $\Pi_1^{0,A}$ set of positive measure is fat.

Our proof of this statement will use the following lemma.

Lemma 4.11 (Kjos-Hanssen 2005). *Let $P, Q \subseteq 2^\omega$ be $\Pi_1^{0,A}$. If PQ is fat, then at least one of P and Q is fat.*

Proof. Let $U = \overline{P}$. Then U is $\Sigma_1^{0,A}$, say $U = \bigcup_{\sigma \in S} N_\sigma$ where $S \subseteq 2^{<\omega}$ is A -r.e. and prefix-free. We have $PQ = P \cup \bigcup_{\sigma \in S} (\sigma \wedge Q)$ where $\sigma \wedge Q = \{\sigma \wedge X \mid X \in Q\}$. Suppose now that PQ is fat, say $R \subseteq PQ$ where R is $\Pi_1^{0,B}$ and $\mu(R) > 0$. By Lemmas 4.7 and 4.8 we may safely assume that $\forall X (X \in R \Rightarrow X$ is B -random), hence $\forall \sigma (R \cap N_\sigma \neq \emptyset \Rightarrow \mu(R \cap N_\sigma) > 0)$. If $R \subseteq P$ then P is fat and we are done. Otherwise there exists $\sigma \in S$ such that $R \cap N_\sigma \neq \emptyset$, hence $\mu(R \cap N_\sigma) > 0$. But $R \cap N_\sigma = R \cap (\sigma \wedge Q) \subseteq \sigma \wedge Q$, hence $\sigma \wedge Q$ is fat, hence Q is fat, Q.E.D. \square

We now prove the implication $1 \Rightarrow 2$ of Theorem 4.6. Assume $A \leq_{LR} B$ and suppose that P is $\Pi_1^{0,A}$ of positive measure. We must show that P is fat.

Let $U = \overline{P}$, so $\mu(U) = 1 - \mu(P) < 1$. Let n be such that $\mu(U)^n < 1/2$. Let $U^k = U \cdots U$ (k times) and $P^k = P \cdots P$ (k times). Note that $U^k = \overline{P^k}$. We have $\mu(U^{nk}) = \mu(U)^{nk} < 1/2^k$, hence U^{nk} , $k = 1, 2, \dots$ is a Martin-Löf test for A -randomness. It follows that $\forall X (X \in \bigcap_k U^k \Rightarrow X$ is not A -random).

By Lemma 4.7 let Q be nonempty $\Pi_1^{0,B}$ such that $\forall X (X \in Q \Rightarrow X$ is B -random). By Lemma 4.8 we have $\forall \sigma (Q \cap N_\sigma \neq \emptyset \Rightarrow \mu(Q \cap N_\sigma) > 0)$.

We claim that $\exists \sigma \exists k (Q \cap N_\sigma \neq \emptyset$ and $Q \cap N_\sigma \cap U^k = \emptyset)$. Otherwise we would have $\forall \sigma \forall k (Q \cap N_\sigma \neq \emptyset \Rightarrow Q \cap N_\sigma \cap U^k \neq \emptyset)$. Therefore, since U^k is $\Sigma_1^{0,A}$, we would be able to find $\sigma_0 \subset \sigma_1 \subset \dots \subset \sigma_k \subset \dots$ such that $N_{\sigma_k} \cap Q \neq \emptyset$ and $N_{\sigma_k} \subseteq U^k$ for all k . Setting $X = \bigcup_k \sigma_k$ we would have $X \in Q$ and $X \in \bigcap_k U^k$. Thus X would be B -random but not A -random, contradicting our assumption $A \leq_{LR} B$. This proves our claim.

Let σ and k be as in our claim. We then have $Q \cap N_\sigma \subseteq P^k$, hence P^k is fat. It follows by Lemma 4.11 that P is fat. This completes the proof of $1 \Rightarrow 2$.

It remains to prove $2 \Rightarrow 3$ and $3 \Rightarrow 4$ and $4 \Rightarrow 1$. The implication $2 \Rightarrow 3$ follows easily from Lemma 4.7. The implication $3 \Rightarrow 4$ is trivial. In order to prove $4 \Rightarrow 1$, we first prove the following lemma due to Kučera [15].

Lemma 4.12 (Kučera 1985). *Let P be $\Pi_1^{0,A}$ of positive measure. Then for all A -random X there exist σ and Y such that $X = \sigma \hat{\ } Y$ and $Y \in P$.*

Proof. Suppose P is $\Pi_1^{0,A}$ of positive measure. As before let $U = \overline{P} = \bigcup_{\sigma \in S} N_\sigma$ where S is A -r.e. and prefix-free. Define U^k and P^k as before. Suppose X is A -random. Since U^k , $k = 1, 2, \dots$ is a Martin-Löf test for A -randomness, we have $X \notin U^k$ for some k . Choosing the least such k , we have $X \in U^{k-1} \cap P^k$, i.e., $X = \sigma_1 \hat{\ } \dots \hat{\ } \sigma_{k-1} \hat{\ } Y$ for some $\sigma_1, \dots, \sigma_{k-1} \in S$ and $Y \in P$. Letting $\sigma = \sigma_1 \hat{\ } \dots \hat{\ } \sigma_{k-1}$ we have $X = \sigma \hat{\ } Y$, Q.E.D. \square

We now prove $4 \Rightarrow 1$. Assuming 4, let Q be $\Pi_1^{0,B}$ of positive measure such that $\forall Y (Y \in Q \Rightarrow Y$ is A -random). To prove 1, suppose X is B -random. Applying Lemma 4.12 with Q and B in place of P and A , we have $X = \sigma \hat{\ } Y$ for some $Y \in Q$. It follows that Y is A -random. Hence X is A -random. Thus $A \leq_{LR} B$, Q.E.D.

This completes the proof of Theorem 4.6. \square

Corollary 4.13 (Kjos-Hanssen 2005). *The binary relation $A \leq_{LR} B$ is Σ_3^0 .*

Proof. Let P_i^C , $i \in \omega$ be a standard, uniform enumeration of all $\Pi_1^{0,C}$ subsets of 2^ω , where C is a Turing oracle. Fix $e \in \omega$ such that for all C , P_e^C is of positive measure¹ and $\forall X (X \in P_e^C \Rightarrow X$ is C -random). By Theorem 4.6, $A \leq_{LR} B$ is equivalent to $\exists i (\mu(P_i^B) > 0$ and $P_i^B \subseteq P_e^A)$. A Tarski/Kuratowski computation (see Rogers [25, Section 14.3]) shows that this is Σ_3^0 . \square

Similarly one can prove:

Corollary 4.14 (Kjos-Hanssen 2005). *The ternary relation $A' \oplus B \leq_{LR} C$ is Σ_3^0 . More generally, for each k the $k+2$ -ary relation $A'_1 \oplus \dots \oplus A'_k \oplus B \leq_{LR} C$ is Σ_3^0 .*

¹For example, we may take $P_e^C = 2^\omega \setminus U_1^C$ where U_n^C , $n = 0, 1, \dots$ is a universal Martin-Löf test for C -randomness as in Theorem 3.2.

Remark 4.15. Earlier Nies had proved that $\{A \mid A \leq_{LR} 0\}$ is Σ_3^0 . Another interesting result due to Nies and Hirschfeldt (see Remark 10.12 below) is that if $A \leq_{LR} 0$ and $B \leq_{LR} 0$ then $A \oplus B \leq_{LR} 0$. Thus $\{A \mid A \leq_{LR} 0\}$ is a Σ_3^0 Turing ideal. See Nies [21, Theorems 6.2 and 7.9].

5 Almost everywhere domination

In this section we exposit some recent results of Kjos-Hanssen/Miller/Solomon [14] concerning almost everywhere domination. We begin by reviewing some earlier definitions and results of Dobrinen/Simpson [7] and Kjos-Hanssen [13].

Definition 5.1 (Dobrinen/Simpson 2003).

1. Let $f, g : \omega \rightarrow \omega$ be total functions. We say that f is *dominated* by g if $f(n) < g(n)$ for all but finitely many n .
2. We say that B is *almost everywhere dominating* if, for measure 1 many $X \in 2^\omega$, each X -recursive function is dominated by some B -recursive function.
3. We say that B is *uniformly almost everywhere dominating* if there is a fixed B -recursive function which dominates all X -recursive functions for measure 1 many $X \in 2^\omega$.

Theorem 5.2 (Dobrinen/Simpson 2003). *The following statements are pairwise equivalent.*

1. B is uniformly almost everywhere dominating.
2. Every Π_2^0 subset of 2^ω includes a $\Sigma_2^{0,B}$ set of the same measure.
3. Every $\Pi_1^{0,0'}$ subset of 2^ω includes a $\Sigma_2^{0,B}$ set of the same measure.

Proof. See Dobrinen/Simpson [7, Theorem 3.2]. □

Theorem 5.3 (Dobrinen/Simpson 2003). *The following statements are pairwise equivalent.*

1. B is almost everywhere dominating.
2. Given a Π_2^0 set $Q \subseteq 2^\omega$, and given $\epsilon > 0$, we can find a $\Pi_1^{0,B}$ set $P \subseteq Q$ such that $\mu(P) \geq \mu(Q) - \epsilon$.
3. Given a $\Pi_1^{0,0'}$ set $Q \subseteq 2^\omega$, and given $\epsilon > 0$, we can find a $\Pi_1^{0,B}$ set $P \subseteq Q$ such that $\mu(P) \geq \mu(Q) - \epsilon$.

Proof. See Dobrinen/Simpson [7, Theorem 3.3]. □

In light of the above results plus Dobrinen/Simpson [7, Conjecture 3.1, Statement 4], the following definition has been made by Kjos-Hanssen [13].

Definition 5.4 (Kjos-Hanssen 2005). We say that B is *positive-measure dominating* if every Π_2^0 subset of 2^ω of positive measure includes a $\Pi_1^{0,B}$ set of positive measure. Equivalently, every $\Pi_1^{0,0'}$ subset of 2^ω of positive measure includes a $\Pi_1^{0,B}$ set of positive measure.

We then have:

Theorem 5.5 (Kjos-Hanssen 2005). B is *positive-measure dominating* if and only if $0' \leq_{LR} B$.

Proof. This is immediate from the special case $A = 0'$ of Theorem 4.6. \square

Corollary 5.6 (Kjos-Hanssen 2005). *The set*

$$\text{PMD} = \{B \mid B \text{ is positive-measure dominating}\}$$

is Σ_3^0 .

Proof. This is immediate from Theorem 5.5 and Corollary 4.14. \square

Our main goal in this section will be to prove the following theorem.

Theorem 5.7 (Kjos-Hanssen/Miller/Solomon 2006). *The following properties of a Turing oracle B are pairwise equivalent.*

1. B is *uniformly almost everywhere dominating*.
2. B is *almost everywhere dominating*.
3. B is *positive-measure dominating*.
4. $0' \leq_{LR} B$.

Remark 5.8. Theorems 5.2, 5.3, 5.5, and 5.7 have implications for the reverse mathematics of measure-theoretic regularity. See Dobrinen/Simpson [7], Binns/Kjos-Hanssen/Lerman/Solomon [2], Cholak/Greenberg/Miller [5], Kjos-Hanssen [13], and Kjos-Hanssen/Miller/Solomon [14]. Namely, it now appears likely that all of the measure-theoretic regularity statements considered by Dobrinen/Simpson [7] are in some sense equivalent.

Corollary 5.9 (Kjos-Hanssen/Miller/Solomon 2006). *The sets*

$$\text{AED} = \{B \mid B \text{ is almost everywhere dominating}\}$$

and

$$\text{UAED} = \{B \mid B \text{ is uniformly almost everywhere dominating}\}$$

are Σ_3^0 . *In fact,* $\text{AED} = \text{UAED} = \text{PMD}$.

Proof. This is immediate from Corollary 5.6 and Theorem 5.7. \square

Remark 5.10. Corollaries 5.6 and 5.9 are of significance for the study of weak degrees (a.k.a., Muchnik degrees) of mass problems associated with nonempty Π_1^0 subsets of 2^ω . This aspect has been examined in Kjos-Hanssen [13] and in Simpson [28]. See also Simpson [27] for some newer results.

We now turn to the proof of Theorem 5.7. The proof (see Remark 5.14 below) will be based on the following lemma and theorem.

Lemma 5.11 (Kjos-Hanssen/Miller/Solomon 2006).

Assume $A \leq_{LR} B$. Let $f : \omega \rightarrow \omega$ be recursive. For all A -r.e. sets I such that $\sum_{i \in I} 1/2^{f(i)} < \infty$, there exists a B -r.e. set $J \supseteq I$ such that $\sum_{i \in J} 1/2^{f(i)} < \infty$.

Proof. We use the following fact from analysis. Given $0 < a_i < 1$, $i = 0, 1, \dots$, we have

$$\sum_{i=0}^{\infty} a_i < \infty \quad \text{if and only if} \quad \prod_{i=0}^{\infty} (1 - a_i) > 0.$$

See for instance Olmstead [23, Theorem III, page 525].

To prove our lemma, let A, B, f, I be as in the hypotheses of the lemma. We may safely assume that $f(i) \neq 0$ for all i . Let

$$D_i = \{X \in 2^\omega \mid \exists n (X(n) = 1 \text{ and } g(i) \leq n < g(i+1))\}$$

where $g(i) = \sum_{k=0}^{i-1} f(k)$. Note that the D_i 's are mutually independent and clopen and $\mu(D_i) = 1 - 1/2^{f(i)}$. Consider the $\Pi_1^{0,A}$ set $P = \bigcap_{i \in I} D_i$. By hypothesis $\sum_{i \in I} 1/2^{f(i)} < \infty$, hence

$$\mu(P) = \prod_{i \in I} \mu(D_i) = \prod_{i \in I} (1 - 1/2^{f(i)}) > 0.$$

Let $Q \subseteq P$ be $\Pi_1^{0,B}$ such that $\mu(Q) > 0$. Let $J = \{i \mid D_i \supseteq Q\}$. Clearly $J \supseteq I$ and J is B -r.e. Moreover $\bigcap_{i \in J} D_i \supseteq Q$, hence

$$\prod_{i \in J} (1 - 1/2^{f(i)}) = \prod_{i \in J} \mu(D_i) = \mu(\bigcap_{i \in J} D_i) \geq \mu(Q) > 0,$$

hence $\sum_{i \in J} 1/2^{f(i)} < \infty$, Q.E.D. \square

Remark 5.12. Under the same hypothesis, $A \leq_{LR} B$, we can obtain the following stronger conclusion. Given a recursive sequence of recursive real numbers $r_i \geq 0$, $i = 0, 1, \dots$, and given an A -r.e. set I such that $\sum_{i \in I} r_i < \infty$, we can effectively find a B -r.e. set $J \supseteq I$ such that $\sum_{i \in J} r_i < \infty$.

Theorem 5.13 (Kjos-Hanssen/Miller/Solomon 2006).

Assume $A \leq_{LR} B$ and $A \leq_T B'$. Then every $\Pi_1^{0,A}$ subset of 2^ω includes a $\Sigma_2^{0,B}$ set of the same measure.

Proof. Assume $A \leq_{LR} B$ and $A \leq_T B'$. We must show that every $\Pi_1^{0,A}$ set includes a $\Sigma_2^{0,B}$ set of the same measure. Equivalently, we shall show that every $\Sigma_1^{0,A}$ set is included a $\Pi_2^{0,B}$ set of the same measure.

Let $U \subseteq 2^\omega$ be $\Sigma_1^{0,A}$. Write

$$U = \{X \mid \exists n (X \upharpoonright n \in S^A)\}$$

where $S^A \subseteq 2^{<\omega}$ is A -r.e. and prefix-free. Let

$$I = \{(\sigma, \tau) \mid \tau \in S^A \text{ with use } \sigma \subset A\}.$$

Thus $U = \{X \mid \exists n \exists \sigma ((\sigma, X \upharpoonright n) \in I)\}$. Clearly I is A -r.e. and

$$\sum_{(\sigma, \tau) \in I} 1/2^{|\tau|} = \sum_{\tau \in S^A} 1/2^{|\tau|} \leq 1.$$

By Lemma 5.11 let $J \supseteq I$ be B -r.e. such that $\sum_{(\sigma, \tau) \in J} 1/2^{|\tau|} < \infty$.

We may safely assume that $A \in 2^\omega$. Since $A \leq_T B'$, let $A = \lim_s A_s$ be a B -recursive approximation of A . Let

$$V_s = \{X \in 2^\omega \mid \exists n \exists \sigma ((\sigma, X \upharpoonright n) \in J_s)\}$$

where

$$J_s = \{(\sigma, \tau) \in J \mid \exists t \geq s (\tau \in S^{A_t} \text{ with use } \sigma \subset A_t)\}.$$

Clearly $\bigcap_s V_s$ is $\Pi_2^{0,B}$. Moreover $I \subseteq \bigcap_s J_s$, hence $U \subseteq \bigcap_s V_s$.

Claim 1: $I = \bigcap_s J_s$.

To see this, suppose $(\sigma, \tau) \notin I$. Then either (1) $\sigma \not\subset A$, or (2) $\sigma \subset A$ but τ is not in S^A with use σ . In case (1) we have $\sigma \not\subset A_t$ for all sufficiently large t , hence $(\sigma, \tau) \notin \bigcap_s J_s$. In case (2) we have $\sigma \subset A_t$ for all sufficiently large t , hence τ is not in S^{A_t} with use σ , hence again $(\sigma, \tau) \notin \bigcap_s J_s$, proving our claim.

Claim 2: $\mu(U) = \mu(\bigcap_s V_s)$.

To see this, fix $\epsilon > 0$. Let F be a finite subset of J such that $\sum_{(\sigma, \tau) \in J \setminus F} 1/2^{|\tau|} < \epsilon$. Let s be so large that, for all $(\sigma, \tau) \in F$, if $(\sigma, \tau) \notin I$ then $(\sigma, \tau) \notin J_s$. Then

$$\mu(V_s \setminus U) \leq \sum_{(\sigma, \tau) \in J_s \setminus I} 1/2^{|\tau|} \leq \sum_{(\sigma, \tau) \in J \setminus F} 1/2^{|\tau|} < \epsilon$$

proving our claim.

This completes the proof of Theorem 5.13. \square

Remark 5.14. Theorem 5.7 is just the special case $A = 0'$ of Theorem 5.13 in light of Theorems 5.2, 5.3, 5.5.

6 Some examples

We have seen in Theorem 5.3 that every Turing oracle $B \geq_T 0'$ is almost everywhere dominating. In this section we construct a $B <_T 0'$ which is almost everywhere dominating. Such examples were first given by Cholak/Greenberg/Miller [5]. We also construct a C such that $0' \leq_{LR} C$ yet $0'$ is not low-for-random relative to C .

To obtain our examples, we combine a construction of Kučera/Terwijn [16] with the Pseudajump Inversion Theorem of Jockusch/Shore [12] and the Join Theorem of Posner/Robinson [24].

Theorem 6.1 (Kučera/Terwijn 1997). *We can find a nonrecursive r.e. set $A \subseteq \omega$ such that $A \leq_{LR} 0$, i.e., A is low-for-random.*

Proof. By Corollary 3.3 we know that $\{X \in 2^\omega \mid X \text{ is random}\}$ is Σ_2^0 and of measure 1. Therefore, we can find a Π_1^0 set $P \subseteq 2^\omega$ such that $\mu(P) > 1/2$ and $\forall X (X \in P \Rightarrow X \text{ is random})$. By uniformly relativizing the construction of P to an arbitrary Turing oracle C , we obtain² a $\Pi_1^{0,C}$ set $P^C \subseteq 2^\omega$ such that $\mu(P^C) > 1/2$ and $\forall X (X \in P^C \Rightarrow X \text{ is } C\text{-random})$. For $C \in 2^\omega$ let $U^C = \overline{P^C} = 2^\omega \setminus P^C$. Note that U^C is uniformly $\Sigma_1^{0,C}$ and $\mu(U^C) < 1/2$.

To prove the theorem, we shall build a nonrecursive r.e. set A and a Σ_1^0 set $V \supseteq U^A$ such that $\mu(V) < 1$. Letting $Q = 2^\omega \setminus V$, it will follow that Q is a Π_1^0 subset of P^A of positive measure. Hence by Theorem 4.6 $A \leq_{LR} 0$, Q.E.D.

It remains to build A and V as specified.

We first establish some notation. Let W_e , $e = 0, 1, \dots$ be a recursive enumeration of the r.e. subsets of ω . Let $W_{e,s}$ be the set of $n \in \omega$ which get into W_e by means of a computation in $\leq s$ steps. Thus $W_e = \bigcup_s W_{e,s}$. Let U_s^C be the set of $X \in 2^\omega$ which get into U^C by means of a computation in $\leq s$ steps using only oracle information from $C \upharpoonright s$. Thus U_s^C , $s = 0, 1, \dots$ is a uniformly C -recursive sequence of clopen sets in 2^ω , and $U^C = \bigcup_s U_s^C$.

Our r.e. set $A \subseteq \omega$ will be constructed as $A = \bigcup_s A_s$ where $A_0 \subseteq A_1 \subseteq \dots \subseteq A_s \subseteq \dots$ is a recursive sequence of finite sets. Let $V = \bigcup_s U_s^{A_s}$. Clearly $U^A \subseteq V \subseteq 2^\omega$ and V is Σ_1^0 . Our construction of A will insure that $\mu(V \setminus U^A) \leq 1/2$. Since $\mu(U^A) < 1/2$, we shall have $\mu(V) < 1$ as desired.

Note that $U_s^{A_s}$, $s = 0, 1, \dots$ is a recursive sequence of clopen sets in 2^ω . Let $V_t = U_t^{A_t} \setminus \bigcup_{s < t} U_s^{A_s}$. Then V_t , $t = 0, 1, \dots$ is a recursive, pairwise disjoint sequence of clopen sets in 2^ω , and $V = \bigcup_s U_s^{A_s} = \bigcup_t V_t$.

Our construction of A is as follows. At stage 0 let $A_0 = \emptyset$. At stage $s + 1$, for each $e < s$ such that $A_s \cap W_{e,s} = \emptyset$, look for $n \in W_{e,s}$ such that $n \geq 2e$ and $\sum_{n < t \leq s} \mu(V_t) \leq 1/2^{e+2}$ and put the least such n into A_{s+1} .

Finally let $A = \bigcup_s A_s$. Clearly A is r.e. By construction, for each e at most one n is put into A for the sake of W_e . Therefore, our condition $n \geq 2e$ insures that $\overline{A} = \omega \setminus A$ is infinite.

Lemma 6.2. *For each e , if W_e is infinite then $A \cap W_e \neq \emptyset$.*

Proof. Since the V_t 's are pairwise disjoint, we have $\sum_t \mu(V_t) = \mu(\bigcup_t V_t) = \mu(V) \leq 1$. Assume that W_e is infinite. Let $n \in W_e$ be so large that $n \geq 2e$ and $\sum_{n < t} \mu(V_t) \leq 1/2^{e+2}$. Let s be so large that $n \in W_{e,s}$. Then by construction $A_{s+1} \cap W_{e,s} \neq \emptyset$. \square

Since \overline{A} is infinite, it follows that A is nonrecursive. (Actually, A is a simple r.e. set. Compare Post's original construction of a simple r.e. set, as exposted in Rogers [25, Section 8.1].)

Lemma 6.3. *We have $\mu(V \setminus U^A) \leq 1/2$.*

²For example, we may take $P^C = 2^\omega \setminus U_2^C$ where U_n^C , $n = 0, 1, 2, \dots$ is a universal Martin-Löf test for C -randomness as in Theorem 3.2.

Proof. Given $X \in V \setminus U^A$, let t be such that $X \in V_t$. Then $X \in U_t^{A_t} \setminus U^A$, hence $A \upharpoonright t \neq A_t \upharpoonright t$. In other words, at some stage $s+1 > t$, some $n < t$ is put into A_{s+1} for the sake of W_e for some $e < s$. For this particular e , the set of such X 's is $\subseteq \bigcup_{n < t \leq s} V_t$, hence of measure $\leq \sum_{n < t \leq s} \mu(V_t) \leq 1/2^{e+2}$. Hence the set of all such X 's is of measure $\leq \sum_e 1/2^{e+2} = 1/2$. \square

This completes the proof of Theorem 6.1. \square

We now present the Pseudojump Inversion Theorem.

Definition 6.4. For an arbitrary Turing oracle C , let W_e^C , $e = 0, 1, \dots$ be a standard, uniform enumeration of all C -r.e. subsets of ω . For each fixed e , the operator $J_e : 2^\omega \rightarrow 2^\omega$ given by $J_e(C) = C \oplus W_e^C$ is called a *pseudojump operator* or an *REA-operator*. (The acronym REA stands for ‘‘r.e. and above’’.)

Remark 6.5. Note that the Turing jump operator $J(C) = C \oplus C'$ is an example of a pseudojump operator. The Pseudojump Inversion Theorem 6.6, due to Jockusch/Shore [12, Theorem 2.1], is a generalization of the Jump Inversion Theorem due to Friedberg (see Rogers [25, Chapter 13, Corollary IX(a)]), replacing the Turing jump operator by an arbitrary pseudojump operator.

Theorem 6.6 (Pseudojump Inversion Theorem). *For any pseudojump operator J_e and any $A \geq_T 0'$, we can find a B such that $J_e(B) \equiv_T B \oplus 0' \equiv_T A$.*

Proof. For $C \in 2^\omega$ and $e \in \omega$, we write $W_{e,s}^C$ to denote the set of $n \in \omega$ which get into W_e^C by means of a computation in $\leq s$ steps using only oracle information from $C \upharpoonright s$. Note that $W_e^C = \bigcup_s W_{e,s}^C$. We also write $W_{e,s}^C = W_{e,s}^\tau$ where $\tau = C \upharpoonright s$.

Given $A \in 2^\omega$, our construction of B is as follows. We define a sequence of strings $\tau_0 \subseteq \tau_1 \subseteq \dots \subseteq \tau_n \subseteq \dots$, $n = 0, 1, \dots$. Let $\tau_0 = \langle \rangle$. For each n , given τ_{2n} , if there exists $\tau \supseteq \tau_{2n}$ such that $n \in W_e^\tau$, let τ_{2n+1} be the least such τ . Otherwise let $\tau_{2n+1} = \tau_{2n}$. For each n let $\tau_{2n+2} = \tau_{2n+1} \hat{\ } \langle A(n) \rangle$. Finally let $B = \bigcup_n \tau_n$. By construction, the sequence $\langle \tau_n \mid n \in \omega \rangle$ is easily seen to be recursive in each of the Turing oracles $J_e(B) = B \oplus W_e^B$ and $B \oplus 0'$ and $A \oplus 0'$. From this, the desired conclusions follow easily. \square

We now use the Pseudojump Inversion Theorem to obtain an example of a $B <_T 0'$ such that B is almost everywhere dominating.

Theorem 6.7. *There exists $B <_T 0'$ such that B is almost everywhere dominating.*

Proof. By uniformly relativizing Theorem 6.1 to an arbitrary Turing oracle C , we obtain a pseudojump operator J_e such that for all C , $J_e(C)$ is $>_T C$ and $\leq_{LR} C$. Applying the Pseudojump Inversion Theorem 6.6 to this operator, we see that for all $A \geq_T 0'$ there exists $B <_T A$ such that $A \equiv_T B \oplus 0'$ and $A \leq_{LR} B$. (Then by Theorem 8.9 we necessarily have $A' \leq_{tt} B'$.) In particular, letting $A = 0'$, we obtain $B <_T 0'$ such that $0' \leq_{LR} B$. (Then by Theorem

8.9 we necessarily have $0'' \leq_{tt} B'$.) By Theorem 5.7 B is almost everywhere dominating. \square

Remark 6.8. In the example of Theorem 6.7, we have $0' \oplus B \equiv_T 0' \leq_{LR} B$, hence $0'$ is low-for-random relative to B . We now use the Join Theorem of Posner/Robinson [24] to obtain a different kind of example, namely a C such that $0' \leq_{LR} C$, hence C is almost everywhere dominating, yet $0'$ is not low-for-random relative to C .

Theorem 6.9 (Join Theorem). *Given A such that $0 <_T A \leq_T 0'$, we can find a C such that $A \oplus C \equiv_T C' \equiv_T 0'$.*

Proof. See Posner/Robinson [24]. \square

Theorem 6.10. *There exists C such that C is almost everywhere dominating yet $0'$ is not low-for-random relative to C .*

Proof. Let B be as in Theorem 6.7, i.e., $B <_T 0'$ and $0' \leq_{LR} B$. Relativizing the Join Theorem 6.9 to B and letting $A = 0'$, we obtain C such that $B \leq_T C$ and $0' \oplus C \equiv_T C' \equiv_T B'$. We then have $0' \leq_{LR} B \leq_T C$, hence $0' \leq_{LR} C$. (It follows that $C' \leq_T 0''$ and $0'' \leq_{tt} C'$.) By Theorem 5.7 C is almost everywhere dominating. We claim that $0' \oplus C \not\leq_{LR} C$, i.e., $0'$ is not low-for-random relative to C . To see this, note that $0' \oplus C \leq_{LR} C$ would imply $(0' \oplus C)' \leq_T C'$ (see Theorem 8.8) which would imply $C'' \leq_T C'$, a contradiction. \square

7 Remarks on Theorem 5.13

This section consists of some technical remarks showing that Theorem 5.13 is, in various senses, best possible.

Remark 7.1. The converse of Theorem 5.13 holds. In other words, if every $\Pi_1^{0,A}$ set includes a $\Sigma_2^{0,B}$ set of the same measure, then $A \leq_{LR} B$ and $A \leq_T B'$.

To see this, assume the conclusion of Theorem 5.13, i.e., every $\Pi_1^{0,A}$ set includes a $\Sigma_2^{0,B}$ set of the same measure. Then $A \leq_{LR} B$ in view of Theorem 4.6. It remains to prove $A \leq_T B'$. Consider the $\Sigma_1^{0,A}$ set $U = \bigcup_{n \in A} O_n$ where

$$O_n = \{X \in 2^\omega \mid \underbrace{\langle 0, \dots, 0, 1 \rangle}_n \subset X\}.$$

Note that $\mu(U) = \sum_{n \in A} 1/2^{n+1}$. Our assumption implies that $U \subseteq Q$ for some $\Pi_2^{0,B}$ set Q such that $\mu(U) = \mu(Q)$. Write $Q = \bigcap_k V_k$ where V_k is uniformly $\Sigma_1^{0,B}$. Then $n \in A$ if and only if $O_n \subseteq Q$, if and only if $\forall k (O_n \subseteq V_k)$, if and only if $\forall k \exists s (O_n \subseteq V_{k,s})$. Thus A is $\Pi_2^{0,B}$. Similarly we can show that $\overline{A} = \omega \setminus A$ is $\Pi_2^{0,B}$. Thus A is $\Delta_2^{0,B}$, i.e., $A \leq_T B'$, Q.E.D.

Remark 7.2. The hypothesis $A \leq_T B'$ cannot be dropped from Theorem 5.13. To see this, recall from Remark 4.4 that there exists a B such that $A \leq_{LR} B$ for

uncountably many A . In particular, there are uncountably many A such that $A \leq_{LR} B$ yet $A \not\leq_T B'$. For such A and B , the conclusion of Theorem 5.13 cannot hold, in view of Remark 7.1.

The following special case of Theorem 5.13 seems interesting in view of Remarks 4.3 and 4.4.

Theorem 7.3. *Assume $A \oplus B \leq_{LR} B$, i.e., A is low-for-random relative to B . Then every $\Pi_1^{0, A \oplus B}$ subset of 2^ω includes a $\Sigma_2^{0, B}$ set of the same measure.*

Proof. We first prove a lemma which is well known. See also Remark 10.12 below.

Lemma 7.4. *If $A \oplus B \leq_{LR} B$ then $A \oplus B \leq_T B'$.*

Proof. Assume $A \oplus B \leq_{LR} B$. By Theorem 10.10 we have $A \oplus B \leq_{LK} B$, i.e., $K^B(\tau) \leq K^{A \oplus B}(\tau) + O(1)$. From this it follows easily that $K^B(A \upharpoonright n) \leq K^B(n) + O(1)$. In other words, using the terminology of Nies [21], A is K -trivial relative to B . By Chaitin's Theorem (see Downey/Hirschfeldt/Nies/Stephan [9, Corollary 6.7(ii)]) relative to B , it follows that A is $\Delta_2^{0, B}$, i.e., $A \leq_T B'$, hence $A \oplus B \leq_T B'$. \square

Theorem 7.3 is now immediate from Theorem 5.13 plus Lemma 7.4. \square

Remark 7.5. Comparison of Theorems 5.13 and 7.3 suggests the following question. If $A \leq_{LR} B$ and $A \leq_T B'$, does it follow that $A \oplus B \leq_{LR} B'$? The answer to this question is negative. Indeed, letting C be as in Theorem 6.10, we have $0' \leq_{LR} C$ and $0'' \leq_{tt} C'$ yet $0' \oplus C \not\leq_{LR} C$.

Question 7.6. If $A \leq_{LR} B$ and $(A \oplus B)' \leq_{tt} B'$, does it follow that $A \oplus B \leq_{LR} B'$? (Compare Theorem 8.9 below.)

8 Superhighness

In this section we present some new results concerning the relationship between LR -reducibility and truth-table reducibility.

Remark 8.1. Recall from Rogers [25, Chapters 8 and 9] the notion of *truth-table reducibility*, denoted \leq_{tt} . A result of Nerode (see Rogers [25, Chapter 9, Theorem XIX]) says that for all $A, B \in 2^\omega$, $A \leq_{tt} B$ if and only if $A = \Phi(B)$ for some total recursive functional $\Phi : 2^\omega \rightarrow 2^\omega$. Thus truth-table reducibility is a special case of Turing reducibility.

Definition 8.2. Let B be a Turing oracle. We say that B is *high* if $0'' \leq_T B'$, i.e., $0''$ is Turing reducible to B' . We say that B is *superhigh* if $0'' \leq_{tt} B'$, i.e., $0''$ is truth-table reducible to B' .

Among other things, we are going to prove that if B is almost everywhere dominating then B is superhigh. We begin with the following definition and lemma.

Definition 8.3. Let A and B be Turing oracles.

1. We say that A is *jump-traceable by B* if for each partial A -recursive function $\psi^A(n)$ there exist recursive functions $f(n)$ and $g(n)$ such that $\forall n (\psi^A(n) \downarrow \Rightarrow \psi^A(n) \in W_{f(n)}^B)$ and $\forall n (|W_{f(n)}^B| \leq g(n))$.
2. We say that A is *weakly jump-traceable by B* if for each partial A -recursive function $\psi^A(n)$ there exists a recursive function $f(n)$ such that $\forall n (\psi^A(n) \downarrow \Rightarrow \psi^A(n) \in W_{f(n)}^B)$ and $\forall n (W_{f(n)}^B \text{ is finite})$.

Lemma 8.4. *If $A \leq_{LR} B$ then A is jump-traceable by B .*

Proof. Consider the A -r.e. set $I = \{(n, m) \mid \psi^A(n) \simeq m\}$. We have

$$\sum_{(n,m) \in I} 1/2^n = \sum_{n \in \text{dom}(\psi^A)} 1/2^n \leq 2 < \infty.$$

By Lemma 5.11 let $J \supseteq I$ be B -r.e. such that $\sum_{(n,m) \in J} 1/2^n < \infty$, say

$$\sum_{(n,m) \in J} 1/2^n \leq 2^c.$$

Let f be a recursive function such that for all n ,

$$W_{f(n)}^B = \{m \mid (n, m) \in J\}.$$

Setting $g(n) = 2^{n+c}$ we obtain the desired conclusions. Note that the bounding function $g(n)$ is not only recursive but primitive recursive. \square

Lemma 8.5. *If A is weakly jump-traceable by B , then $A' \leq_T A \oplus B'$.*

Proof. Consider the partial A -recursive function $\psi^A(e) \simeq$ least s such that $\varphi_{e,s}^{(1),A}(e) \downarrow$. Note that $A' = \text{dom}(\psi^A)$. By weak jump-traceability, let $f(e)$ be recursive such that $\forall e (\psi^A(e) \downarrow \Rightarrow \psi^A(e) \in W_{f(e)}^B)$ and $\forall e (W_{f(e)}^B \text{ is finite})$. Consider the total function $h(e) = \max(W_{f(e)}^B \cup \{0\})$. Note that $h(e)$ is recursive relative to B' , and $e \in A'$ if and only if $\varphi_{e,h(e)}^{(1),A}(e) \downarrow$. Thus $A' \leq_T A \oplus B'$. \square

Lemma 8.6. *If A is jump-traceable by B and B -r.e., then $A' \leq_{tt} B'$.*

Proof. Since A is B -r.e., let $A = \bigcup_s A_s$ where A_s , $s = 0, 1, \dots$ is a B -recursive sequence of finite sets with $A_0 \subseteq A_1 \subseteq \dots \subseteq A_s \subseteq A_{s+1} \subseteq \dots$. We identify the sets A and A_s with their characteristic functions. Consider the partial A -recursive function $\theta^A(e) \simeq$ the least $\sigma \subset A$ such that $\varphi_{e,|\sigma|}^{(1),\sigma}(e) \downarrow$. Clearly $A' = \text{dom}(\theta^A) = \{e \mid \theta^A(e) \downarrow\}$. By jump-traceability let $f(e)$ and $g(e)$ be recursive functions such that $\forall e (\theta^A(e) \downarrow \Rightarrow \theta^A(e) \in W_{f(e)}^B)$ and $\forall e (|W_{f(e)}^B| \leq g(e))$. We may safely assume that each $\sigma \in W_{f(e)}^B$ satisfies $\varphi_{e,|\sigma|}^{(1),\sigma}(e) \downarrow$. For all e and all $i < g(e)$ let $\sigma_{e,i} \simeq$ the i th member of $W_{f(e)}^B$ in order of B -recursive enumeration. We may now compute A' from B' as follows. Given e , for each $i < g(e)$ ask the B' oracle whether $\exists s (\sigma_{e,i} \downarrow \subset A_s)$ and whether $\exists s \exists t (s < t \text{ and } \sigma_{e,i} \downarrow \subset A_s \text{ and } \sigma_{e,i} \not\subset A_t)$. Upon receiving the answers to these $2g(e)$ -many questions, we know the finite set $F_e = \{i < g(e) \mid \sigma_{e,i} \downarrow \subset A\}$. Then $e \in A'$ if and only if F_e is nonempty. Thus $A' \leq_{tt} B'$, Q.E.D. \square

Remark 8.7. In the proof of Lemma 8.6 we have $\varphi_e^{(1),A}(e) \simeq \varphi_{e,|\sigma_{e,i}|}^{(1),\sigma_{e,i}}(e)$ for any $i \in F_e$. Thus we see that, under the hypotheses of Lemma 8.6, the function

$$h^A(e) = \begin{cases} \varphi_e^{(1),A}(e) + 1 & \text{if } \varphi_e^{(1),A}(e) \downarrow, \\ 0 & \text{otherwise} \end{cases}$$

is B' -recursive with recursively bounded use of B' and unbounded use of B . This conclusion is in general strictly stronger than either of the conditions $A' \leq_{tt} B'$ and A jump-traceable by B . However, it is equivalent to their conjunction if $A \geq_T B$, and it is equivalent to each of them individually if $A \geq_T B$ and A is B -r.e.

Theorem 8.8. *If $A \leq_{LR} B$ then $A' \leq_T A \oplus B'$.*

Proof. This is immediate from Lemmas 8.4 and 8.5. □

Theorem 8.9. *If $A \leq_{LR} B$ and A is B -r.e., then $A' \leq_{tt} B'$.*

Proof. This is immediate from Lemmas 8.4 and 8.6. □

Theorem 8.10. *If $0' \leq_{LR} B$ then $0'' \leq_{tt} B'$.*

Proof. This is the special case $A = 0'$ of Theorem 8.9. □

Theorem 8.11. *If B is almost everywhere dominating, then B is superhigh.*

Proof. This is a restatement of Theorem 8.10 in light of Theorem 5.7. □

Remark 8.12. Our Theorems 8.8, 8.9, 8.10, 8.11 are closely related to some results of Nies [21, 22]. Nies proved that if $A \leq_{LR} 0$ then A is *superlow*, i.e., $A' \leq_{tt} 0'$ (see Nies [21, Corollary 7.6]). Relativizing the arguments of Nies to a Turing oracle B , one can show that if $A \oplus B \leq_{LR} B$ then $(A \oplus B)' \leq_{tt} B'$. (See also Remark 10.12 below.) In particular, if $0' \oplus B \leq_{LR} B$ then $0'' \leq_{tt} B'$. Our Theorem 8.10 improves this by weakening the hypothesis $0' \oplus B \leq_{LR} B$ to $0' \leq_{LR} B$. In addition, Nies [21, paragraph preceding Proposition 8.3] announced that if A and B are r.e. and $A \leq_{LR} B$ then $A' \leq_{tt} B'$. Our Theorem 8.9 improves this by eliminating the hypothesis that B is r.e.

Question 8.13. If $A \leq_{LR} B$ and $A \leq_T 0'$, does it follow that $A' \leq_{tt} B'$?

9 Counterexamples via duality

In the previous section we have seen that if B is almost everywhere dominating then B is superhigh, which implies that B is high. In this section we present counterexamples showing that neither of these implications can be reversed.

In order to obtain our counterexamples, we use a duality technique which has been used previously by Jockusch/Shore [12], Mohrherr [20], Nies [21], and Kjos-Hanssen [13]. The technique is based on the following theorem due to Jockusch/Shore [12] which we call the Duality Theorem.

Theorem 9.1 (Duality Theorem). *Given a pseudojump operator J_e , we can find an r.e. set B such that $J_e(B) \equiv_T 0'$.*

Proof. See Jockusch/Shore [12, Theorem 3.1]. □

We shall see that the Duality Theorem provides a powerful method of passing from “lowness properties” to “highness properties” as in Table 1. The meaning of Table 1 is that, if we have a uniformly relativizable construction of an r.e. set A with some but not all of the properties on the left side of the table, then we can apply the Duality Theorem to obtain an r.e. set B with corresponding properties on the right side of the table.

$\leq_T 0$	$\geq_T 0'$
low	high
superlow	superhigh
$\leq_{LR} 0$	$\geq_{LR} 0'$
low-for-random	a. e. dominating

Table 1: Duality

As our first application of the Duality Theorem, we note the following improvement of the counterexample in Theorem 6.7. A similar result was first obtained by Cholak/Greenberg/Miller [5] using a different method.

Theorem 9.2 (Cholak/Greenberg/Miller 2004). *There exists an r.e. set B which is $<_T 0'$ and almost everywhere dominating.*

Proof. In Theorem 6.1 we have constructed an r.e. set A which is $>_T 0$ and $\leq_{LR} 0$. By uniformly relativizing this construction to an arbitrary Turing oracle C , we obtain a pseudojump operator J_e such that for all C , $J_e(C)$ is $>_T C$ and $\leq_{LR} C$. Applying the Duality Theorem 9.1 to this operator, we obtain an r.e. set B which is $<_T 0'$ and $\geq_{LR} 0'$. It follows by Theorems 5.7 and 8.11 that B is almost everywhere dominating and therefore superhigh. Examples of this kind were first obtained by Cholak/Greenberg/Miller [5, Section 2]. See also Binns/Kjos-Hanssen/Lerman/Solomon [2]. □

We now apply the Duality Theorem to obtain additional counterexamples.

Lemma 9.3. *There exists an r.e. set which is superlow and not low-for-random.*

Proof. We omit the proof, which is fairly straightforward. □

Theorem 9.4. *There exists an r.e. set B which is superhigh and not almost everywhere dominating.*

Proof. By uniformly relativizing the proof of Lemma 9.3, we obtain a pseudojump operator J_e such that for all C , $J_e(C)$ is superlow relative to C and not

low-for-random relative to C . In other words, $J_e(C)' \leq_{tt} C'$ and $J_e(C) \not\leq_{LR} C$. Applying the Duality Theorem 9.1, we obtain an r.e. set B such that $0'' \leq_{tt} B'$ and $0' \not\leq_{LR} B$. It follows by Theorem 5.7 that B is superhigh and not almost everywhere dominating. \square

The next lemma is due to Bickford/Mills and Mohrherr [20].

Lemma 9.5. *There exists an r.e. set A which is low and not superlow.*

Proof. See Mohrherr [20, Theorem 3] and Bickford/Mills (reference in Mohrherr [20]). \square

The following counterexample is due to Mohrherr [20].

Theorem 9.6. *There exists an r.e. set B which is high and not superhigh.*

Proof. We argue as in Mohrherr [20, Theorem 5]. By uniformly relativizing the proof of Lemma 9.5, we obtain a pseudojump operator J_e such that for all C , $J_e(C)$ is low and not superlow relative to C . In other words, $J_e(C)' \leq_T C'$ and $\not\leq_{tt} C'$. Applying the Duality Theorem 9.1, we obtain an r.e. set B such that $0''$ is $\leq_T B'$ and $\not\leq_{tt} B'$. In other words, B is high and not superhigh. \square

10 Prefix-free Kolmogorov complexity

In our exposition of Martin-Löf randomness and LR -reducibility in Sections 3 through 8 above, we have followed the effective measure-theoretic and descriptive set-theoretic approach due to Kučera [15]. The purpose of this section is to explain an alternative approach in terms of relativized prefix-free Kolmogorov complexity. This approach is the one which has been followed by Downey/Hirschfeldt/Nies/Stephan [9] and Nies [21].

Lemma 10.1. *Given a recursive sequence of natural numbers m_i , $i = 0, 1, \dots$ such that $\sum_{i=0}^{\infty} 1/2^{m_i} \leq 1$, we can effectively find a recursive, one-to-one, prefix-free sequence of strings σ_i , $i = 0, 1, \dots$ such that $|\sigma_i| = m_i$ for all i .*

Proof. Assume inductively that we have chosen σ_i , $0 \leq i < k$. Assume also that we have chosen another finite set of strings, D_k . Define a *partition* to be a finite, maximal, prefix-free set of strings. We start with $D_0 = \{\langle \rangle\}$ and assume inductively that D_k has the following properties:

- (a) $D_k \cap \{\sigma_i \mid 0 \leq i < k\} = \emptyset$.
- (b) $D_k \cup \{\sigma_i \mid 0 \leq i < k\}$ is a partition.
- (c) The strings in D_k are all of different lengths. I.e., for all $\rho, \sigma \in D_k$, if $\rho \neq \sigma$ then $|\rho| \neq |\sigma|$.

In fact we shall have a stronger property: $\forall \rho, \sigma \in D_k (\rho <_{\text{lex}} \sigma \Rightarrow |\rho| > |\sigma|)$, where $<_{\text{lex}}$ denotes the lexicographical order.

We claim that $m_k \geq \min\{|\rho| \mid \rho \in D_k\}$.
 Otherwise $m_k < \min\{|\rho| \mid \rho \in D_k\}$, hence by (c) we have

$$\frac{1}{2^{m_k}} > \sum_{\rho \in D_k} \frac{1}{2^{|\rho|}},$$

hence by (b) we have

$$\sum_{i=0}^k \frac{1}{2^{m_i}} = \frac{1}{2^{m_k}} + \sum_{i=0}^{k-1} \frac{1}{2^{m_i}} > \sum_{\rho \in D_k} \frac{1}{2^{|\rho|}} + \sum_{i=0}^{k-1} \frac{1}{2^{|\sigma_i|}} = 1,$$

a contradiction.

By the above claim, let $\rho_k \in D_k$ be such that $|\rho_k| \leq m_k$ and $|\rho_k|$ is as large as possible. Let

$$\sigma_k = \rho_k \hat{\ } \underbrace{(0, \dots, 0)}_{m_k - |\rho_k|}.$$

Then $|\sigma_k| = m_k$ and $\sigma_k \not\subseteq \sigma_i$, $\sigma_i \not\subseteq \sigma_k$ for $0 \leq i < k$. Let

$$D_{k+1} = D_k \setminus \{\rho_k\} \cup \{\rho_k \hat{\ } \underbrace{(0, \dots, 0, 1)}_j \mid 0 \leq j < m_k - |\rho_k|\}.$$

It is easy to verify that (a), (b), (c) hold with $k+1$ in place of k . \square

Definition 10.2. We define a *machine* to be a partial recursive function from $2^{<\omega}$ into $2^{<\omega}$. A machine M is said to be *prefix-free* if its domain

$$\text{dom}(M) = \{\sigma \in 2^{<\omega} \mid M(\sigma) \downarrow\}$$

is prefix-free.

If M is a prefix-free machine, then clearly $\sum_{M(\sigma) \downarrow} 1/2^{|\sigma|} \leq 1$, and this is known as the *Kraft Inequality*. Conversely we have the following theorem attributed by Nies [21, Theorem 3.2] and Downey/Hirschfeldt/Nies/Stephan [9, Theorem 2.1] to Chaitin [3].

Theorem 10.3 (Kraft/Chaitin Theorem). *Let L be an r.e. subset of $\omega \times 2^{<\omega}$ such that $\sum_{(m,\tau) \in L} 1/2^m \leq 1$. Then we can effectively find a prefix-free machine M such that for all $(m, \tau) \in L$ there exists σ such that $|\sigma| = m$ and $M(\sigma) \simeq \tau$.*

Proof. Let (m_i, τ_i) , $i = 0, 1, \dots$ be a one-to-one recursive enumeration of L . By Lemma 10.1 let σ_i , $i = 0, 1, \dots$ be a recursive, prefix-free sequence of strings such that $|\sigma_i| = m_i$ for all i . Define $M(\sigma_i) = \tau_i$ for all i . \square

Remark 10.4. In the Kraft/Chaitin Theorem 10.3, the pairs $(m, \tau) \in L$ are sometimes called *axioms* for the prefix-free machine M . The idea of the theorem is that we can construct a prefix-free machine given only an abstract description of the machine in terms of axioms.

Definition 10.5 (prefix-free Kolmogorov complexity). A prefix-free machine U is said to be *universal* if for all prefix-free machines M there exists a string ρ such that for all strings σ , $M(\sigma) \simeq U(\rho \hat{\ } \sigma)$. The *prefix-free Kolmogorov complexity* of a string τ is defined to be

$$K(\tau) = \min\{|\sigma| \mid U(\sigma) \simeq \tau\}$$

where U is a universal prefix-free machine. Note that $K(\tau)$ is well defined up to within $O(1)$. In other words, if we let

$$\widehat{K}(\tau) = \min\{|\sigma| \mid \widehat{U}(\sigma) \simeq \tau\}$$

where \widehat{U} is another universal prefix-free machine, then $|K(\tau) - \widehat{K}(\tau)| \leq O(1)$, i.e., $\exists c \forall \tau (|K(\tau) - \widehat{K}(\tau)| \leq c)$.

Corollary 10.6. *Let L be an r.e. subset of $\omega \times 2^{<\omega}$ such that $\sum_{(m,\tau) \in L} 1/2^m < \infty$. Then for all $(m, \tau) \in L$ we have $K(\tau) \leq m + O(1)$.*

Proof. Let c be such that $\sum_{(m,\tau) \in L} 1/2^m \leq 2^c$. Then $\sum_{(m,\tau) \in L} 1/2^{m+c} \leq 1$, so by the Kraft/Chaitin Theorem 10.3 let M be a prefix-free machine such that for all $(m, \tau) \in L$ there exists σ such that $|\sigma| = m + c$ and $M(\sigma) = \tau$. Let ρ be such that $M(\sigma) \simeq U(\rho \hat{\ } \sigma)$ for all σ . Then for all $(m, \tau) \in L$ we have $K(\tau) \leq m + c + |\rho|$. This completes the proof. \square

The following theorem has been attributed by Chaitin [3, Remark following Theorem 4.2] and Nies [21, Section 1] to Claus Peter Schnorr.

Theorem 10.7 (Schnorr's Theorem). *Given $X \in 2^\omega$, we have that X is random if and only if $K(X \upharpoonright n) \geq n - O(1)$, i.e., $\exists c \forall n (K(X \upharpoonright n) \geq n - c)$.*

Proof. For $c = 1, 2, \dots$ let $V_c = \{X \mid \exists n (K(X \upharpoonright n) < n - c)\}$. Note that $V_c \subseteq 2^\omega$ is uniformly Σ_1^0 . We claim that $\mu(V_c) < 1/2^c$. To see this, for each τ such that $K(\tau) < |\tau| - c$ choose σ such that $U(\sigma) = \tau$ and $|\sigma| < |\tau| - c$. Here U is a universal prefix-free machine. By the Kraft Inequality we have

$$\sum_{\sigma} \frac{1}{2^{|\sigma|}} \leq 1$$

hence

$$\mu(V_c) \leq \sum_{\tau} \frac{1}{2^{|\tau|}} < \sum_{\sigma} \frac{1}{2^{|\sigma|+c}} \leq \frac{1}{2^c}$$

thus proving the claim. We now see that V_c , $c = 1, 2, \dots$ is a Martin-Löf test. Therefore, if X is random, we have $\exists c (X \notin V_c)$, i.e., $\exists c \forall n (K(X \upharpoonright n) \geq n - c)$, i.e., $K(X \upharpoonright n) \geq n - O(1)$. This is one direction of the theorem.

For the converse, assume X is not random, say $X \in \bigcap_n U_n$ where $U_n \subseteq 2^\omega$ is uniformly Σ_1^0 and $\mu(U_n) \leq 1/2^n$, $n = 1, 2, \dots$. Let T_n be uniformly r.e. and prefix-free such that $U_n = \bigcup_{\tau \in T_n} N_\tau$. We have

$$\sum_n \sum_{\tau \in T_{2n}} \frac{1}{2^{|\tau|-n}} = \sum_n 2^n \mu(U_{2n}) \leq \sum_n \frac{2^n}{2^{2n}} = \sum_n \frac{1}{2^n} = 1.$$

Hence by Corollary 10.6, $K(\tau) \leq |\tau| - n + O(1)$ for all n and all $\tau \in T_{2n}$. Since $X \in U_{2n}$ for all n , it follows that $K(X \upharpoonright n) \not\geq n - O(1)$. \square

Remark 10.8. The above definitions and results can be uniformly relativized to an arbitrary Turing oracle C . Thus, a *prefix-free C -machine* is a partial C -recursive function M from $2^{<\omega}$ to $2^{<\omega}$ such that $\text{dom}(M)$ is prefix-free. We define $K^C(\tau) = \min\{|\sigma| \mid U^C(\sigma) \simeq \tau\}$ where U^C is a universal prefix-free C -machine. The relativization of Schnorr's Theorem says that $X \in 2^\omega$ is C -random if and only if $K^C(X \upharpoonright n) \geq n - O(1)$.

Definition 10.9 (Nies 2002). Let A and B be Turing oracles. We say that A is *LK-reducible* to B , abbreviated $A \leq_{LK} B$, if $K^B(\tau) \leq K^A(\tau) + O(1)$. In other words, for some constant c we have $K^B(\tau) \leq K^A(\tau) + c$ for all strings τ .

Theorem 10.10 (Kjos-Hanssen/Miller/Solomon 2006).
The following statements are pairwise equivalent.

1. $A \leq_{LK} B$.
2. $A \leq_{LR} B$.
3. For any A -r.e. set $I \subseteq \omega \times \omega$ such that $\sum_{(m,n) \in I} 1/2^m < \infty$, there exists a B -r.e. set $J \supseteq I$ such that $\sum_{(m,n) \in J} 1/2^m < \infty$.

Proof. The implication $1 \Rightarrow 2$ follows from Schnorr's Theorem relativized to A and B . The implication $2 \Rightarrow 3$ follows from Lemma 5.11. Now assume 3 and let $I = \{(m, \tau) \mid K^A(\tau) \leq m\}$. Clearly I is A -r.e. and $\sum_{(m,\tau) \in I} 1/2^m < 2 < \infty$, so let $J \supseteq I$ be B -r.e. such that $\sum_{(m,\tau) \in J} 1/2^m < \infty$. By Corollary 10.6 relative to B we have $K^B(\tau) \leq m + O(1)$ for all $(m, \tau) \in J$. Since $I \subseteq J$ it follows that $K^B(\tau) \leq K^A(\tau) + O(1)$ for all τ . This proves $3 \Rightarrow 1$. \square

Remark 10.11. The reducibilities \leq_{LK} and \leq_{LR} were originally defined by Nies [21, Section 8]. In the direction of Theorem 10.10, Nies proved the equivalence $A \leq_{LK} 0 \iff A \leq_{LR} 0$ (see [21, Corollary 5.3]) and noted that $A \leq_{LK} B$ implies $A \leq_{LR} B$. The full equivalence $A \leq_{LK} B \iff A \leq_{LR} B$ is due to Kjos-Hanssen/Miller/Solomon [14].

Remark 10.12. By means of relativized prefix-free Kolmogorov complexity, one can prove a number of interesting results concerning \leq_{LR} for which no other proofs are presently known. Among these results are:

1. If $A \leq_{LR} 0$ and $B \leq_{LR} 0$ then $A \oplus B \leq_{LR} 0$. (See Nies [21, Theorem 6.2] and Downey/Hirschfeldt/Nies/Stephan [9, Theorem 7.2].) More generally, if $A \oplus C \leq_{LR} C$ and $B \oplus C \leq_{LR} C$ then $A \oplus B \oplus C \leq_{LR} C$.

Caution: It is not in general true that if $A \leq_{LR} C$ and $B \leq_{LR} C$ then $A \oplus B \leq_{LR} C$. Indeed, in Theorem 6.10 above we have constructed a C such that $\theta' \leq_{LR} C$ yet $\theta' \oplus C \not\leq_{LR} C$.

2. If $A \leq_{LR} 0$ then $A \leq_T 0'$, in fact $A' \leq_{tt} 0'$ (see Nies [21, Corollary 7.6]). More generally, if $A \oplus B \leq_{LR} B$ then $A \leq_T B'$ (see Lemma 7.4 above), in fact $(A \oplus B)' \leq_{tt} B'$.
3. If $A \leq_{LR} 0$ then we can find an r.e. set D such that $D \leq_{LR} 0$ and $A \leq_T D$, in fact $A \leq_{tt} D$. (See Nies [21, Theorems 7.4 and 6.2].) More generally, if $A \oplus B \leq_{LR} B$ then we can find a B -r.e. set D such that $D \oplus B \leq_{LR} B$ and $A \leq_T D$, in fact $A \leq_{tt} D$. See also Theorem 8.9 above.
4. If $A \equiv_{LR} B$ then $A \oplus B \leq_{LR} A$ and $A \oplus B \leq_{LR} B$ (see Nies [21, Proposition 8.3(iii)]), hence $A \leq_T B'$ and $B \leq_T A'$, in fact $A' \equiv_{tt} B' \equiv_{tt} (A \oplus B)'$.

In particular, for each B there are only countably many A such that $A \equiv_{LR} B$.

Caution: By Miller/Yu [19, 18], there exists a B such that $\{A \mid A \leq_{LR} B\}$ is uncountable. In fact, Barmpalias/Lewis/Soskova [1] have shown that this holds for any B which is generalized superhigh.

The presently available proofs of most of these results use not only relativized prefix-free Kolmogorov complexity but also the formidable “golden run” machinery of Nies [21, Section 6]. It would be desirable to find alternative proofs in the style of Sections 3–8 above.

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