

# Medvedev degrees of 2-dimensional subshifts of finite type

Stephen G. Simpson\*  
Department of Mathematics  
Pennsylvania State University  
<http://www.math.psu.edu/simpson>  
[simpson@math.psu.edu](mailto:simpson@math.psu.edu)

Submitted: May 1, 2007  
This draft: January 16, 2009

## Abstract

In this paper we apply some fundamental concepts and results from recursion theory in order to obtain an apparently new counterexample in symbolic dynamics. Two sets  $X$  and  $Y$  are said to be *Medvedev equivalent* if there exist partial recursive functionals from  $X$  into  $Y$  and vice versa. The *Medvedev degree* of  $X$  is the equivalence class of  $X$  under Medvedev equivalence. There is an extensive recursion-theoretic literature on the lattice of Medvedev degrees of nonempty  $\Pi_1^0$  subsets of  $\{0, 1\}^{\mathbb{N}}$ . This lattice is known as  $\mathcal{P}_s$ . We prove that  $\mathcal{P}_s$  consists precisely of the Medvedev degrees of 2-dimensional subshifts of finite type. We use this result to obtain an infinite collection of 2-dimensional subshifts of finite type which are, in a certain sense, mutually incompatible.

**Definition 1.** Let  $A$  be a finite set of symbols. The *full 2-dimensional shift* on  $A$  is the dynamical system consisting of the natural action of  $\mathbb{Z}^2$  on the compact set  $A^{\mathbb{Z}^2}$ . A *2-dimensional subshift* is a nonempty closed set  $X \subseteq A^{\mathbb{Z}^2}$  which is invariant under the action of  $\mathbb{Z}^2$ . A 2-dimensional subshift  $X$  is said to be *of finite type* if it is defined by a finite set of forbidden configurations. An interesting paper on 2-dimensional subshifts of finite type is Mozes [22]. A standard reference for the 1-dimensional case is the book of Lind/Marcus [20], which also includes an appendix [20, §13.10] on the 2-dimensional case.

---

\*The author's research was partially supported by the United States National Science Foundation, grant DMS-0600823, and by the Cada and Susan Grove Mathematics Enhancement Endowment at the Pennsylvania State University. In addition, the author thanks Ali Dashti for telling him of the results in Cenzer/Dashti/King [9] and Benjamin Weiss for calling his attention to the papers of Hochman/Meyerovitch [17] and Mozes [22].

**Remark 1.** In the study of 2-dimensional subshifts of finite type, it has been useful to note that they are essentially the same thing as *tiling problems* in the sense of Wang [41].

On the one hand, if  $F$  is a finite set of Wang tiles, let  $T_F$  be the set of tilings of the plane by  $F$ . Identifying  $T_F$  as a subset of  $F^{\mathbb{Z}^2}$ , it is clear that  $T_F$  is a 2-dimensional subshift of finite type. Conversely, given a 2-dimensional subshift of finite type,  $X$ , it is easy to construct a finite set of Wang tiles,  $F$ , such that  $T_F$  is recursively isomorphic to  $X$ . Namely, let  $r$  be a positive integer which is greater than or equal to the diameters of all of the forbidden configurations defining  $X$ , and let  $F \subseteq A^{\{1, \dots, r\}^2}$  be the set of  $r \times r$  configurations which do not contain any of the forbidden configurations. Our isomorphism of  $X$  onto  $T_F$  associates to each point  $x \in X$  a tiling  $t \in T_F$  defined by  $t(m, n)(i, j) = x(m + i, n + j)$  for all  $(m, n) \in \mathbb{Z}^2$  and  $(i, j) \in \{1, \dots, r\}^2$ . The adjacency rules for  $\tau_1, \tau_2 \in F$  are: (a)  $\tau_1$  is allowed to occur immediately to the left of  $\tau_2$  if and only if  $\tau_1(i + 1, j) = \tau_2(i, j)$  for all  $1 \leq i \leq r - 1$  and  $1 \leq j \leq r$ , and (b)  $\tau_1$  is allowed to occur immediately below  $\tau_2$  if and only if  $\tau_1(i, j + 1) = \tau_2(i, j)$  for all  $1 \leq i \leq r$  and  $1 \leq j \leq r - 1$ .

Mozes [22, 23] and recently Hochman/Meyerovitch [17] have used the tiling methods of Wang [41] and Robinson [26] to construct 2-dimensional subshifts of finite type with interesting dynamical properties.

**Definition 2.** If  $X$  is a 2-dimensional subshift, the *shift operators*  $S_1, S_2 : X \rightarrow X$  are defined by  $S_1(x)(m, n) = x(m + 1, n)$  and  $S_2(x)(m, n) = x(m, n + 1)$  for all  $x \in X$  and  $(m, n) \in \mathbb{Z}^2$ . If  $X$  and  $Y$  are 2-dimensional subshifts, a *shift morphism* of  $X$  into  $Y$  is a continuous function  $f : X \rightarrow Y$  which commutes with the shift operators, i.e.,  $f(S_1(x)) = S_1(f(x))$  and  $f(S_2(x)) = S_2(f(x))$  for all  $x \in X$ . It follows that  $f$  commutes with the action of  $\mathbb{Z}^2$  on  $X$  and  $Y$ . A *shift isomorphism* of  $X$  onto  $Y$  is a shift morphism of  $X$  one-to-one onto  $Y$ , i.e., a homeomorphism of  $X$  onto  $Y$  which commutes with  $S_1$  and  $S_2$ . We say that  $X$  and  $Y$  are *shift isomorphic* if there exists a shift isomorphism of  $X$  onto  $Y$ .

**Definition 3.** Recall that two sets  $X$  and  $Y$  are *Medvedev equivalent* (see Rogers [27, §13.7] and Medvedev [21]) if there exist partial recursive functionals which map  $X$  into  $Y$  and  $Y$  into  $X$ . Also,  $X$  and  $Y$  are *recursively homeomorphic* if there exists a partial recursive functional which maps  $X$  one-to-one onto  $Y$  whose inverse is a partial recursive functional which maps  $Y$  one-to-one onto  $X$ . Finally,  $X$  and  $Y$  are *Muchnik equivalent* (see Muchnik [24]) if (a) each point in  $X$  is carried by some partial recursive functional to some point in  $Y$ , and (b) each point in  $Y$  is carried by some partial recursive functional to some point in  $X$ . Obviously recursive homeomorphism implies Medvedev equivalence, and Medvedev equivalence implies Muchnik equivalence. However, the converses do not hold. The equivalence classes under Medvedev equivalence and Muchnik equivalence are known as *Medvedev degrees* and *Muchnik degrees* respectively.

**Remark 2.** We shall now show that the Medvedev degree, Muchnik degree, and recursive homeomorphism type of a 2-dimensional subshift  $X$  depend only on the shift isomorphism type of  $X$ .

**Theorem 1.** *All shift morphisms  $f : X \rightarrow Y$  are given by partial recursive functionals. If  $X$  and  $Y$  are shift isomorphic, then  $X$  and  $Y$  are recursively homeomorphic, hence Medvedev equivalent, hence Muchnik equivalent.*

*Proof.* Let  $A$  and  $B$  be finite sets of symbols such that  $X$  and  $Y$  are subshifts of the full 2-dimensional shifts  $A^{\mathbb{Z}^2}$  and  $B^{\mathbb{Z}^2}$  respectively. Let  $f : X \rightarrow Y$  be a shift morphism. By the Curtis/Hedlund/Lyndon Theorem (see Boyle [8] or Lind/Marcus [20])  $f$  is a *block code*, i.e., we can find an integer  $r \geq 0$  and a projection  $\pi : A^{\{-r, \dots, r\}^2} \rightarrow B$  such that

$$f(x)(m, n) = \pi(S_1^m S_2^n(x) \upharpoonright \{-r, \dots, r\}^2)$$

for all  $x \in X$  and  $(m, n) \in \mathbb{Z}^2$ . In particular  $f$  is given by a partial recursive functional. If moreover  $f$  is a shift isomorphism of  $X$  onto  $Y$ , it follows that  $f$  is a recursive homeomorphism of  $X$  onto  $Y$ . This completes the proof.  $\square$

**Definition 4.** A closed set  $P$  is said to be  $\Pi_1^0$  (see Rogers [27, Chapter 15]) if it is *effectively closed*, i.e.,  $P$  is the complement of the union of a recursive sequence of basic open neighborhoods. Let  $\mathcal{P}_s$  (respectively  $\mathcal{P}_w$ ) be the lattice of Medvedev degrees (respectively Muchnik degrees) of nonempty  $\Pi_1^0$  sets  $P \subseteq \{0, 1\}^{\mathbb{N}}$ . It is known that the lattices  $\mathcal{P}_s$  and  $\mathcal{P}_w$  are distributive. There is a lattice homomorphism of  $\mathcal{P}_s$  onto  $\mathcal{P}_w$  obtained by mapping the Medvedev degree of  $P$  to the Muchnik degree of  $P$ . The lattices  $\mathcal{P}_s$  and  $\mathcal{P}_w$  are mathematically rich and have been studied extensively. See Alfeld [1], Binns [2, 3, 4, 5], Binns/Simpson [6], Cenzer/Hinman [10], Cole/Simpson [12], and Simpson [29, 30, 31, 33, 34, 35, 36, 37, 38, 39].

**Remark 3.** An open problem is to characterize the recursive homeomorphism types of 2-dimensional subshifts of finite type. Obviously every 2-dimensional subshift of finite type is a  $\Pi_1^0$  subset of  $A^{\mathbb{Z}^2}$  and is therefore recursively homeomorphic to, hence of the same Medvedev degree and Muchnik degree as, a  $\Pi_1^0$  subset of  $\{0, 1\}^{\mathbb{N}}$ . We shall now prove a theorem in the opposite direction. Namely, every  $\Pi_1^0$  subset of  $\{0, 1\}^{\mathbb{N}}$  is of the same Medvedev degree as, hence of the same Muchnik degree as, some 2-dimensional subshift of finite type. This characterization of the Medvedev degrees and Muchnik degrees of 2-dimensional subshifts of finite type appears to be new.

**Theorem 2.** *Given a nonempty  $\Pi_1^0$  set  $P \subseteq \{0, 1\}^{\mathbb{N}}$ , we can find a 2-dimensional subshift of finite type which is of the same Medvedev degree as  $P$ .*

*Proof.* Our proof uses the tiling constructions of Robinson [26] and Hanf [16] and Myers [25].

Let  $F$  be a finite set of Wang tiles, and let  $\tau \in F$  be a distinguished tile in  $F$ . We write  $T_F^\tau = \{t \in T_F \mid t(0, 0) = \tau\}$ . The tilings in  $T_F^\tau$  are said to be *origin-constrained*. Given a  $\Pi_1^0$  set  $P \subseteq \{0, 1\}^{\mathbb{N}}$ , Hanf [16] shows how to construct an origin-constrained tiling system  $F, \tau$  and a projection  $\pi : F \rightarrow \{0, 1\}$  such that

$$P = \{\{\pi(t(n, 0)) \mid n \in \mathbb{N}\} \mid t \in T_F^\tau\}.$$

Namely,  $F$  and  $\tau$  are such that each origin-constrained tiling  $t \in T_F^\tau$  describes a non-halting run of a particular deterministic Turing machine  $M$  starting with

$$\langle \pi(t(n, 0)) \mid n \in \mathbb{N} \rangle \in \{0, 1\}^{\mathbb{N}}$$

inscribed on the right half of the Turing machine tape. The distinguished tile  $\tau$  represents the initial internal state of  $M$ . Since  $M$  is deterministic,  $t$  is uniformly recursive relative to  $\langle \pi(t(n, 0)) \mid n \in \mathbb{N} \rangle$ . It follows that  $T_F^\tau$  and  $P$  are recursively homeomorphic.

Now, starting with  $F$  and  $\tau$  as above, Myers [25] shows how to construct a finite set of tiles  $F'$  and a projection  $\pi' : F' \rightarrow F$  with the following properties:

1.  $F'$  is a *Robinson set of tiles*.  
(Unexplained terms are defined in Myers' paper [25].)
2. For each tiling  $t' \in T_{F'}$ , the  $\pi'$ -images of the *center rows* of the *finite boards* of  $t'$  are *synchronized*.
3. For each tiling  $t' \in T_{F'}$ , the result of *piecing together* the  $\pi'$ -images of the center rows of the finite boards of  $t'$  is  $\langle t(n, 0) \mid n \in \mathbb{Z} \rangle$  for some  $t \in T_F^\tau$ . It follows that  $t$  is uniformly recursive relative to  $t'$ .
4. Given an origin-constrained tiling  $t \in T_F^\tau$ , we can find a tiling  $t' \in T_{F'}$  such that  $\langle t(n, 0) \mid n \in \mathbb{Z} \rangle$  is the result of piecing together the  $\pi'$ -images of the center rows of the finite boards of  $t'$ . Moreover  $t'$  is uniformly recursive relative to  $t$ .

Properties 3 and 4 imply that  $T_{F'}$  is Medvedev equivalent to  $T_F^\tau$ . Therefore,  $T_{F'}$  is Medvedev equivalent to  $P$ . Since  $T_{F'}$  is a 2-dimensional subshift of finite type, our theorem is proved.  $\square$

**Remark 4.** Actually Hanf [16] and Myers [25] deal only with  $\Pi_1^0$  sets  $P \subseteq \{0, 1\}^{\mathbb{N}}$  of the form

$$P = S(I, J) = \{p \in \{0, 1\}^{\mathbb{N}} \mid p \text{ separates } I, J\}$$

where  $I$  and  $J$  are recursively enumerable subsets of  $\mathbb{N}$ . However, their constructions work just as well for arbitrary  $\Pi_1^0$  sets  $P \subseteq \{0, 1\}^{\mathbb{N}}$ .

**Remark 5.** Leonid A. Levin has kindly pointed out that our Theorem 2 was already implicit in his remark [19, last paragraph of Section 3] concerning tilings of the plane with a 2-adic coordinate system.

**Definition 5.** Let  $\mathcal{S}^2$  be the set of all 2-dimensional subshifts. For  $X, Y \in \mathcal{S}^2$  we write  $X \geq Y$  if there exists a shift morphism  $f : X \rightarrow Y$ . Obviously  $\geq$  is transitive and reflexive. We write  $X \equiv Y$  if  $X \geq Y$  and  $Y \geq X$ . Obviously  $\equiv$  is an equivalence relation on  $\mathcal{S}^2$ . Obviously  $X \equiv Y$  whenever  $X$  and  $Y$  are shift isomorphic, but the converse does not hold. Let  $\mathcal{S}^2/\equiv$  be the set of equivalence classes of  $\mathcal{S}^2$  modulo  $\equiv$ . There is an obvious partial ordering of  $\mathcal{S}^2/\equiv$  induced by  $\geq$ . It is easy to verify that, under this partial ordering,  $\mathcal{S}^2/\equiv$  is a distributive lattice. Let  $\mathcal{S}_{\text{fin}}^2$  be the subset of  $\mathcal{S}^2$  consisting of the 2-dimensional subshifts of finite type. It is easy to verify that  $\mathcal{S}_{\text{fin}}^2/\equiv$  is a sublattice of  $\mathcal{S}^2/\equiv$ .

**Theorem 3.** *There is a natural lattice homomorphism of  $\mathcal{S}_{\text{fin}}^2/\equiv$  onto  $\mathcal{P}_s$ . Namely, for each  $X \in \mathcal{S}_{\text{fin}}^2$  we map the  $\equiv$ -equivalence class of  $X$  to the Medvedev degree of  $X$ .*

*Proof.* The greatest lower bound and least upper bound operations in each of the lattices  $\mathcal{S}^2/\equiv$ ,  $\mathcal{S}_{\text{fin}}^2/\equiv$ ,  $\mathcal{P}_s$ ,  $\mathcal{P}_w$  are given by disjoint union and Cartesian product respectively. In particular, our mapping of  $\mathcal{S}_{\text{fin}}^2/\equiv$  to  $\mathcal{P}_s$  is a lattice homomorphism. By Theorem 2 this homomorphism is onto  $\mathcal{P}_s$ .  $\square$

**Remark 6.** A consequence of Theorem 3 is that there is a natural lattice homomorphism of  $\mathcal{S}_{\text{fin}}^2/\equiv$  onto  $\mathcal{P}_w$ , obtained by mapping the  $\equiv$ -equivalence class of  $X \in \mathcal{S}_{\text{fin}}^2$  to the Muchnik degree of  $X$ . In particular we have a new invariant, the Muchnik degree, associated to (shift isomorphism types of) 2-dimensional subshifts of finite type. Obviously this invariant is qualitatively different from other such invariants which have been considered previously in the symbolic dynamics literature. Moreover, this new invariant is of clear interest, inasmuch as  $\mathcal{P}_w$  is known to contain many specific, natural Muchnik degrees which are closely related to significant topics in the foundations of mathematics. Among these topics are algorithmic randomness [36, 38], almost everywhere domination [39], reverse mathematics [32, 36], the hyperarithmetical hierarchy [12], and Kolmogorov complexity [18].

**Remark 7.** We shall now present an application which is stated purely in terms of 2-dimensional subshifts, with no reference to Medvedev degrees or Muchnik degrees. Namely, we shall construct an infinite collection of 2-dimensional subshifts of finite type which are, in a certain sense, mutually incompatible.

**Definition 6.** If  $X$  and  $Y$  are 2-dimensional subshifts on  $k$  and  $l$  symbols respectively, let  $X+Y$  and  $X \times Y$  be the disjoint union and Cartesian product of  $X$  and  $Y$ . These are 2-dimensional subshifts on  $k+l$  and  $kl$  symbols respectively. If  $X = (X, S_1, S_2)$  is a 2-dimensional subshift on  $k$  symbols, and if  $a, b, c, d$  are integers such that  $ad - bc \neq 0$ , let  $X[a, b, c, d] = (X, S_1^a S_2^b, S_1^c S_2^d)$ . This is a 2-dimensional subshift on  $k^{|ad-bc|}$  symbols.

**Definition 7.** If  $\mathcal{U}$  is a collection of 2-dimensional subshifts, let  $\text{cl}(\mathcal{U})$  be the closure of  $\mathcal{U}$  under the operations of Definition 6. In other words,  $\text{cl}(\mathcal{U})$  is the smallest collection of 2-dimensional subshifts with the following properties:

1. For all  $X \in \mathcal{U}$ ,  $X \in \text{cl}(\mathcal{U})$ .
2. For all  $X \in \text{cl}(\mathcal{U})$  and  $Y \in \text{cl}(\mathcal{U})$ ,  $X + Y \in \text{cl}(\mathcal{U})$  and  $X \times Y \in \text{cl}(\mathcal{U})$ .
3. For all  $X \in \text{cl}(\mathcal{U})$  and all  $a, b, c, d \in \mathbb{Z}$  with  $ad - bc \neq 0$ ,  $X[a, b, c, d] \in \text{cl}(\mathcal{U})$ .

**Theorem 4.** *We can find an infinite collection of 2-dimensional subshifts of finite type,  $\mathcal{W}$ , such that for all partitions of  $\mathcal{W}$  into two subcollections,  $\mathcal{U}$  and  $\mathcal{V}$ , there is no shift morphism of  $X$  into  $Y$  for any  $X \in \text{cl}(\mathcal{U})$  and  $Y \in \text{cl}(\mathcal{V})$ .*

*Proof.* By Binns/Simpson [6] let  $P_i$ ,  $i = 1, 2, \dots$ , be nonempty  $\Pi_1^0$  subsets of  $2^{\mathbb{N}}$  whose Medvedev degrees are *independent*, i.e.,  $P_i + \dots + P_{i_m}$  is not Medvedev

reducible to  $P_{j_1} \times \dots \times P_{j_n}$  provided  $\{i_1, \dots, i_m\} \cap \{j_1, \dots, j_n\} = \emptyset$ . By Theorem 2, for each  $i = 1, 2, \dots$  let  $X_i$  be a 2-dimensional subshift of finite type which is Medvedev equivalent to  $P_i$ . Let  $\mathcal{W}$  be the collection  $X_i$ ,  $i = 1, 2, \dots$ , and let  $\mathcal{U}, \mathcal{V}$  be a partition of  $\mathcal{W}$ . By induction on  $X \in \text{cl}(\mathcal{U})$  and  $Y \in \text{cl}(\mathcal{V})$  we can easily show that neither of  $X$  and  $Y$  is Medvedev reducible to the other. Hence by Theorem 1 there is no shift morphism of  $X$  into  $Y$  or vice versa. Q.E.D.  $\square$

**Remark 8.** In this paper we have dealt only with 2-dimensional subshifts. What about the 1-dimensional case? Clearly Theorem 1 holds in this case. On the other hand, Theorems 2 and 3 fail, because all 1-dimensional subshifts of finite type contain periodic points and are therefore of Medvedev degree zero and of Muchnik degree zero. An open problem is to characterize the Medvedev degrees, Muchnik degrees, and recursive homeomorphism types of 1-dimensional  $\Pi_1^0$  subshifts. In this direction Cenzer/Dashti/King [9] have constructed a 1-dimensional  $\Pi_1^0$  subshift which is of nonzero Muchnik degree, hence of nonzero Medvedev degree. We do not know whether Theorems 2, 3, and 4 hold for 1-dimensional  $\Pi_1^0$  subshifts. We do not know whether Theorem 4 holds for 1-dimensional subshifts of finite type, or for 1-dimensional subshifts in general.

## References

- [1] Christopher Alfeld. Non-branching degrees in the Medvedev lattice of  $\Pi_1^0$  classes. *Journal of Symbolic Logic*, 72:81–97, 2007.
- [2] Stephen Binns. *The Medvedev and Muchnik Lattices of  $\Pi_1^0$  Classes*. PhD thesis, Pennsylvania State University, August 2003. VII + 80 pages.
- [3] Stephen Binns. A splitting theorem for the Medvedev and Muchnik lattices. *Mathematical Logic Quarterly*, 49:327–335, 2003.
- [4] Stephen Binns. Small  $\Pi_1^0$  classes. *Archive for Mathematical Logic*, 45:393–410, 2006.
- [5] Stephen Binns. Hyperimmunity in  $2^{\mathbb{N}}$ . *Notre Dame Journal of Formal Logic*, 48:293–316, 2007.
- [6] Stephen Binns and Stephen G. Simpson. Embeddings into the Medvedev and Muchnik lattices of  $\Pi_1^0$  classes. *Archive for Mathematical Logic*, 43:399–414, 2004.
- [7] F. Blanchard, A. Maass, and A. Nogueira, editors. *Topics in Symbolic Dynamics and Applications (Temuco, 1997)*. Number 279 in London Mathematical Society Lecture Notes Series. Cambridge University Press, 2000. XVI + 245 pages.
- [8] Mike Boyle. Algebraic aspects of symbolic dynamics. In [7], pages 57–88, 2000.

- [9] Douglas Cenzer, S. Ali Dashti, and Jonathan L. F. King. Effective symbolic dynamics. 13 March 2007. Preprint, 12 pages.
- [10] Douglas Cenzer and Peter G. Hinman. Density of the Medvedev lattice of  $\Pi_1^0$  classes. *Archive for Mathematical Logic*, 42:583–600, 2003.
- [11] C.-T. Chong, Q. Feng, T. A. Slaman, W. H. Woodin, and Y. Yang, editors. *Computational Prospects of Infinity: Proceedings of the Logic Workshop at the Institute for Mathematical Sciences, June 20 – August 15, 2005*. Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore. World Scientific Publishing Company, Ltd., 2007. In preparation, to appear.
- [12] Joshua A. Cole and Stephen G. Simpson. Mass problems and hyperarithmeticity. Preprint, 20 pages, 28 November 2006, submitted for publication.
- [13] COMP-THY e-mail list. <http://listserv.nd.edu/archives/comp-thy.html>, September 1995 to the present.
- [14] *FOCS 2002: The 43rd Annual IEEE Symposium on Foundations of Computer Science*. IEEE Computer Society, 2002. XII + 811 pages.
- [15] FOM e-mail list. <http://www.cs.nyu.edu/mailman/listinfo/fom/>, September 1997 to the present.
- [16] William Hanf. Nonrecursive tilings of the plane, I. *Journal of Symbolic Logic*, 39:283–285, 1974.
- [17] Michael Hochman and Tom Meyerovitch. A characterization of the entropies of multidimensional shifts of finite type. 7 March 2007. Preprint, arXiv:math:DS/0703206v1, 27 pages.
- [18] Bjørn Kjos-Hanssen and Stephen G. Simpson. Mass problems and Kolmogorov complexity. Preprint, 1 page, 4 October 2006, in preparation.
- [19] Leonid A. Levin. Forbidden information. In [14], pages 761–768, 2002.
- [20] Douglas Lind and Brian Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, 1995. XVI + 495 pages.
- [21] Yuri T. Medvedev. Degrees of difficulty of mass problems. *Doklady Akademii Nauk SSSR, n.s.*, 104:501–504, 1955. In Russian.
- [22] Shahar Mozes. Tilings, substitution systems, and the dynamical systems generated by them. *Journal d'Analyse Mathématique*, 53:139–186, 1989.
- [23] Shahar Mozes. A zero entropy, mixing of all orders tiling system. In [40], pages 319–325. 1992.
- [24] Albert A. Muchnik. On strong and weak reducibilities of algorithmic problems. *Sibirskii Matematicheskii Zhurnal*, 4:1328–1341, 1963. In Russian.

- [25] Dale Myers. Nonrecursive tilings of the plane, II. *Journal of Symbolic Logic*, 39:286–294, 1974.
- [26] Raphael M. Robinson. Undecidability and nonperiodicity of tilings of the plane. *Inventiones Mathematicae*, 12:177–209, 1971.
- [27] Hartley Rogers, Jr. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, 1967. XIX + 482 pages.
- [28] S. G. Simpson, editor. *Reverse Mathematics 2001*. Number 21 in Lecture Notes in Logic. Association for Symbolic Logic, 2005. X + 401 pages.
- [29] Stephen G. Simpson. Some fundamental issues concerning degrees of unsolvability. In [11]. Preprint, 15 December 2005, 21 pages, to appear.
- [30] Stephen G. Simpson. FOM: natural r.e. degrees;  $\Pi_1$  classes. FOM e-mail list [15], 13 August 1999.
- [31] Stephen G. Simpson. FOM: priority arguments; Kleene-r.e. degrees;  $\Pi_1$  classes. FOM e-mail list [15], 16 August 1999.
- [32] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Perspectives in Mathematical Logic. Springer-Verlag, 1999. XIV + 445 pages.
- [33] Stephen G. Simpson. Medvedev degrees of nonempty  $\Pi_1^0$  subsets of  $2^\omega$ . COMP-THY e-mail list [13], 9 June 2000.
- [34] Stephen G. Simpson. Mass problems. 24 May 2004. Preprint, 24 pages, submitted for publication.
- [35] Stephen G. Simpson. FOM: natural r.e. degrees. FOM e-mail list [15], 27 February 2005.
- [36] Stephen G. Simpson. Mass problems and randomness. *Bulletin of Symbolic Logic*, 11:1–27, 2005.
- [37] Stephen G. Simpson.  $\Pi_1^0$  sets and models of  $WKL_0$ . In [28], pages 352–378. 2005.
- [38] Stephen G. Simpson. An extension of the recursively enumerable Turing degrees. *Journal of the London Mathematical Society*, 75:287–297, 2007.
- [39] Stephen G. Simpson. Mass problems and almost everywhere domination. *Mathematical Logic Quarterly*, 53:483–492, 2007.
- [40] P. Walters, editor. *Symbolic Dynamics and its Applications*. Number 135 in Contemporary Mathematics. American Mathematical Society, 1992. XVI + 451 pages.
- [41] Hao Wang. Proving theorems by pattern recognition, II. *Bell System Technical Journal*, 40:1–42, 1961.