

# Math 597B – Homework #6

April 20, 2007

A set  $U \subseteq 2^{\mathbb{N}}$  is said to be *dense* if  $U \cap N_\sigma \neq \emptyset$  for all  $\sigma \in 2^{<\mathbb{N}}$ . A point  $X \in 2^{\mathbb{N}}$  is said to be *weakly generic* if  $X \in U$  for all dense  $\Sigma_1^0$  sets  $U \subseteq 2^{\mathbb{N}}$ . A point  $X \in 2^{\mathbb{N}}$  is said to be *weakly random* if  $X \notin P$  for all  $\Pi_1^0$  sets  $P \subseteq 2^{\mathbb{N}}$  of measure 0.

- (10 points) The concept of weak genericity was used in class to solve some of the problems which were in the Homework #4 problem set. Specifically, we sketched proofs of the following facts:

- The set  $\{X \in 2^{\mathbb{N}} \mid X \text{ is weakly generic}\}$  is dense.
- If  $X$  is weakly generic, then  $X$  is weakly random.
- If  $X$  is weakly generic, then  $X$  does not obey the Strong Law of Large Numbers, i.e.,

$$\lim_{n \rightarrow \infty} \frac{|\{m < n \mid X(m) = 1\}|}{n} \neq \frac{1}{2}.$$

In fact, the limit in question does not exist, because the lim sup is 1 and the lim inf is 0.

- We can find  $X, Y \in 2^{\mathbb{N}}$  such that  $X \oplus Y$  is weakly generic yet  $X \equiv_T Y$ .

Please work out the detailed proofs of these facts.

- (10 points) Given a point  $X \in 2^{\mathbb{N}}$ , we write  $\dim(X) =$  the *effective Hausdorff dimension* of  $X$ , defined by

$$\dim(X) = \liminf_{n \rightarrow \infty} \frac{K(X \upharpoonright n)}{n} = \liminf_{n \rightarrow \infty} \frac{C(X \upharpoonright n)}{n}.$$

Prove the following:

- For all  $X$  we have  $0 \leq \dim(X) \leq 1$ .
  - For all  $r$  in the interval  $0 \leq r \leq 1$ , there exists  $X$  such that  $\dim(X) = r$ .
  - If  $X$  is random, then  $\dim(X) = 1$ .
  - If  $X$  is weakly generic, then  $\dim(X) = 0$ .
- (5 points) Define  $X$  to be *2-random* if  $X \notin \bigcap_n U_n$  whenever  $U_n \subseteq 2^{\mathbb{N}}$  is uniformly  $\Sigma_2^0$  with  $\mu(U_n) \leq 1/2^n$  for all  $n$ .

Prove that  $X$  is 2-random if and only if  $X$  is random relative to  $0'$ .

- (5 points) Prove that if  $X$  is 2-random then  $X' \equiv_T X \oplus 0'$ .
- (5 points) In Problem 4, what if we assume only that  $X$  is strongly random?