

# Math 597B – Homework #3

February 23, 2007

Recall that  $\vee$  and  $\wedge$  denote, respectively, *join* (i.e., the least upper bound operator) and *meet* (i.e., the greatest lower bound operator) in the partial ordering of Turing degrees. A Turing degree  $\mathbf{c}$  is said to be *join-reducible* (respectively *meet-reducible*) if there exist Turing degrees  $\mathbf{a} \neq \mathbf{c}$  and  $\mathbf{b} \neq \mathbf{c}$  such that  $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$  (respectively  $\mathbf{a} \wedge \mathbf{b} = \mathbf{c}$ ).

Note: For an arbitrary pair of Turing degrees  $\mathbf{a} = \deg_T(f)$  and  $\mathbf{b} = \deg_T(g)$ , we always have  $\mathbf{a} \vee \mathbf{b} = \deg_T(f \oplus g)$ . On the other hand,  $\mathbf{a} \wedge \mathbf{b}$  may or may not exist. See Problem 7 below.

1. Prove that  $\mathbf{0}$  is meet-reducible.
2. By relativizing Problem 1, prove that every Turing degree is meet-reducible.
3. Prove that  $\mathbf{0}'$  is join-reducible.
4. By relativizing Problem 3 and applying the Friedberg Jump Theorem, prove that every Turing degree  $\mathbf{c} \geq \mathbf{0}'$  is join-reducible.

Note: It can be shown that not every Turing degree  $\mathbf{c} > \mathbf{0}$  is join-reducible.

5. Prove the following theorem.

Given an ascending sequence of Turing degrees

$$\mathbf{c}_1 < \mathbf{c}_2 < \cdots < \mathbf{c}_n < \cdots$$

we can find a pair of Turing degrees  $\mathbf{a}, \mathbf{b}$  such that for all  $\mathbf{d}$ ,  
 $\exists n (\mathbf{d} \leq \mathbf{c}_n)$  if and only if  $\mathbf{d} \leq \mathbf{a}$  and  $\mathbf{d} \leq \mathbf{b}$ .

6. Use the result of Problem 5 to prove that no ascending sequence of Turing degrees has a least upper bound.
7. For any pair of Turing degrees  $\mathbf{a}, \mathbf{b}$  as in Problem 5, prove that  $\mathbf{a} \wedge \mathbf{b}$  does not exist.