Math 597B – Homework #1

February 2, 2007

Definitions:

For $A, B \subseteq \mathbb{N}$ we say $A \leq_m B$ (A is *many-one reducible* to B) if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $\forall x\,(x \in A \Leftrightarrow f(x) \in B)$.

We say $A \equiv_m B$ (A is *many-one equivalent* to B) if $A \leq_m B$ and $B \leq_m A$.

Define $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

1. (a) Show that $\leq_m$ is reflexive and transitive.
   (b) Show that $\equiv_m$ is an equivalence relation.
   (c) Show that $\oplus$ acts as a least upper bound operator for $\leq_m$. That is, $A \oplus B \leq_m C$ if and only if $A \leq_m C$ and $B \leq_m C$.

2. Define

   (a) $K = \{x \mid \varphi_x^{(1)}(x)\downarrow\}$, diagonal halting problem.
   (b) $H = \{x \mid \varphi_x^{(1)}(0)\downarrow\}$, the halting problem.
   (c) $G = \{3^x 5^y 7^z \mid \varphi_x^{(1)}(y) \simeq z\}$.
   (d) $E = \{x \mid \varphi_x^{(1)}$ is the empty function$\}$.
   (e) $R = \{3^x 5^y \mid y \in \text{range}(\varphi_x^{(1)})\}$.

   Show that $H \equiv_m K \equiv_m G \equiv_m \overline{E} \equiv_m R$.

   Here we are writing $\overline{A} = \mathbb{N} \setminus A =$ the complement of A.

3. (Rice's Theorem) Let $\mathcal{P}$ be the class of 1-place partial computable functions. For any subclass $\mathcal{C} \subseteq \mathcal{P}$, define $I_{\mathcal{C}} = \{x \mid \varphi_x^{(1)} \in \mathcal{C}\}$. This is called an *index set*. Prove that $I_{\mathcal{C}}$ is always noncomputable, except in the trivial cases $\mathcal{C} = \emptyset$ and $\mathcal{C} = \mathcal{P}$.

4. Write a register machine program which computes the *exponential function*, i.e., the 2-place number-theoretic function $\exp(x, y) = x^y$. Note that $x^0 = 1$ for all x.