

Math 558 – Midterm Exam #3

April 11, 2007

(4 problems)

1. Let I be a (possibly infinite) index set. Let $\langle X_i \rangle_{i \in I}$ and $\langle Y_i \rangle_{i \in I}$ be families of sets, both indexed by I . Assume that there exists a function with domain $\bigcup_{i \in I} X_i$ and range $\prod_{i \in I} Y_i$. Prove that for at least one $i \in I$ there exists a function with domain X_i and range Y_i .
2. True or false.
 - (a) There exists a 1-1 function with domain \mathbb{N} and range $\mathbb{Q} \times \mathbb{Q}$.
 - (b) There exists a 1-1 function with domain \mathbb{R}^2 and range \mathbb{R}^3 . (Here \mathbb{R} denotes the set of real numbers.)
 - (c) Every function $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ may be identified with an appropriately chosen, pure, well-founded set.
 - (d) There is a set consisting of all well-founded relational structures.
 - (e) If R_1 and R_2 are two different well-orderings of the same set A , then the relational structures (A, R_1) and (A, R_2) have the same order type.
 - (f) For all uncountable cardinals κ we have $(\aleph_2^{\aleph_1})^\kappa = 2^\kappa$.
 - (g) The cofinality of 2^{\aleph_3} is $\leq \aleph_3$.
 - (h) For all ordinals α the cardinal $\aleph_{\alpha+1}$ is regular.
 - (i) For all ordinals α and β we have $\alpha \cdot \beta = \beta \cdot \alpha$.
 - (j) For all von Neumann ordinals α we have $\alpha + 1 = \alpha \cup \{\alpha\}$ and $\alpha + 2 = \alpha \cup \{\alpha, \alpha + 1\}$.

3. Let X be a set, all of whose elements are sets.

By a *chain within X* we mean a set $C \subseteq X$ such that for all $u, v \in C$ either $u \subseteq v$ or $v \subseteq u$. A chain within X is said to be *maximal* if it is not properly included in any other chain within X .

Use the Axiom of Choice plus transfinite recursion to prove that there exists a maximal chain within X .

Note: This is a version of Zorn's Lemma.

4. A *transitive set* is defined to be a set Y such that every set which is an element of Y is also a subset of Y .
 - (a) Prove that each von Neumann ordinal is a transitive set.
 - (b) Given a set X , prove that there exists a unique set Y with the following properties:
 - (i) Y is a transitive set.
 - (ii) Y includes X .
 - (iii) Y is the smallest transitive set which includes X .

Note: This set Y is called the *transitive closure* of X , abbreviated $\text{TC}(X)$. Thus X is transitive if and only if $X = \text{TC}(X)$.