Math 558 – Midterm Exam #2

March 9, 2007

(5 problems)

1. Write a sentence in the language $+, -, \cdot, 0, 1, <, =$ which, when interpreted in the real number system $\mathbb{R}$, expresses the following proposition of Euclidean plane geometry:

   > Given a circle C and a point P other than the center of C, among all points on C there is exactly one which is at minimum distance from P.

2. Find a quantifier-free formula in the language $+, -, \cdot, 0, 1, <, =$ which is equivalent, over the real number system $\mathbb{R}$, to the formula

$$\exists x \, (ax^3 + bx^2 + cx + d > 0) \, .$$

3. Find a pair of natural numbers $r, a \in \mathbb{N}$ such that $\beta(r, a, 0) = 2$, $\beta(r, a, 1) = 3$, and $\beta(r, a, 2) = 8$.

4. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

   For each $k \in \mathbb{N}$ let

$$f^k = \underbrace{f \circ \cdots \circ f}_{k},$$

   i.e., $f^0(n) = n$ and $f^{k+1}(n) = f(f^k(n))$.

   Write a sentence in the language $+, \cdot, 0, 1, =$ which, when interpreted in the natural number system $\mathbb{N}$, expresses the statement that for all $n \geq 1$ there exists k such that $f^k(n) = 1$.

5. Does there exist a constant c such that the following holds?

   > Given a formula $F(x)$ in the language $+, \cdot, 0, 1, =$ with exactly one free variable x, we can find a formula $F^*(x)$ in the same language which is equivalent to $F(x)$ over the natural number system $\mathbb{N}$, and which contains at most c quantifiers.

   Prove your answer.