Math 558, Foundations of Mathematics

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Some Homework Problems

1. Show that the function $f = \lambda n\,[\,n\text{th digit of } \sqrt{2}\,]$ is primitive recursive.

   Solution: Using the bounded least number operator, define a primitive recursive function

   $$\begin{aligned} g(n) &= \lfloor \sqrt{2} \cdot 10^n \rfloor \\ &= \text{least } x \text{ such that } \left(\frac{x+1}{10^n}\right)^2 > 2 \\ & \text{ or, equivalently, } (x+1)^2 > 2 \cdot 10^{2n} \\ &= \text{least } x < 2 \cdot 10^n \text{ such that } (x+1)^2 > 2 \cdot 10^{2n}\,; \end{aligned}$$

   thus $g(0) = 1$, $g(1) = 14$, $g(2) = 141$, $g(3) = 1414$, $g(4) = 14142$, etc. It follows that

   $$f(n) = \text{Remainder}(g(n), 10)$$

   is also primitive recursive.

2. Exercise on the Ackermann function.
   (a) Show that each F_n is primitive recursive.
   (b) i. $F_n(x) > 0$.
       ii. $F_n(x+1) > F_n(x)$, *i.e.* F_n is monotone.
       iii. $F_n(x) > x$.
       iv. $F_{n+1} \geq F_n(x+1)$.

(c) Every primitive recursive function is dominated by F_n for some n.

(d) $\lambda x[F_x(x)]$ is not primitive recursive.

Solution. Recall the definition

$$\begin{array}{rcl} F_0(x) & = & x+1 \\ F_{n+1}(x) & = & \underbrace{F_n \cdots F_n}_{x+1}(1)\,. \end{array}$$

(a) By induction on n. Obviously F_0 is primitive recursive. The recursion equations

$$\begin{array}{rcl} F_{n+1}(0) & = & F_n(1) \\ F_{n+1}(x+1) & = & F_n(F_{n+1}(x)) \end{array}$$

show that if F_n is primitive recursive then so is F_{n+1}.

(b) $F_n(x) > 0$ is easily proved by induction on n. Prove F_n monotone and $F_n(x) > x$ simultaneously by induction on n: $F_n(0) > 0$ and monotonicity imply $F_n(x) > x$ for all x, hence $F_{n+1}(x+1) = F_n(F_{n+1}(x)) > F_{n+1}(x)$. Prove $F_{n+1}(x) \geq F_n(x+1)$ by induction on x: For $x = 0$ we have $F_{n+1}(0) = F_n(1)$, and for $x+1$ we have $F_{n+1}(x+1) = F_n(F_{n+1}(x)) \geq F_n(F_n(x+1))$ by inductive hypothesis, and $F_n(F_n(x+1)) \geq F_n(x+2)$ since $F_n(x+1) \geq x+2$ and F_n is monotone.

(c) Recall that $f(x_1, \ldots, x_k)$ is said to be dominated by F_n if

$$f(x_1, \ldots, x_k) \leq F_n(\max(x_1, \ldots, x_k))$$

for all $x_1, \ldots, x_k$. Clearly the initial functions are dominated by F_0. For composition, if $f = h(g_1, \ldots, g_m)$, let n be sufficiently large so that F_{n+1} dominates $g_1, \ldots, g_m$ and F_n dominates h. Then an easy computation shows that F_{n+1} dominates f. For primitive recursion, if f is obtained by primitive recursion from g and h, let n be sufficiently large so that F_{n+1} dominates g and F_n dominates h. We claim that

$$f(y, x_1, \ldots, x_k) \leq F_{n+1}(y + \max(x_1, \ldots, x_k))$$

for all $x_1, \ldots, x_k, y$. This is easily proved by induction on y. Note also that

$$F_{n+2}(z) = F_{n+1} \underbrace{F_{n+1} \cdots F_{n+1}}_{z}(1) \geq F_{n+1}(2z+1)$$

for all z, since $F_{n+1}(w) \geq F_1(w) = w + 2$ for all w. Thus we have

$$\begin{array}{rcl} f(y, x_1, \ldots, x_k) & \leq & F_{n+1}(y + \max(x_1, \ldots, x_k)) \\ & \leq & F_{n+1}(2 \max(y, x_1, \ldots, x_k)) \\ & \leq & F_{n+2}(\max(y, x_1, \ldots, x_k)) \end{array}$$

and this completes the proof.

(d) If $\lambda x[F_x(x)]$ were primitive recursive, then $\lambda x[F_x(x) + 1]$ would be primitive recursive, hence dominated by F_n for some n, in particular $F_n(n) + 1 \leq F_n(n)$, a contradiction.

3. Show that every computable function can be computed by a register machine with 4 registers.

4. Show that a total recursive function $f(x_1, \ldots, x_k)$ is primitive recursive if and only if there exists an index e of f such that

$$\lambda x_1 \cdots x_k \, [\, \mathrm{Stop}(e, x_1, \ldots, x_k) \,]$$

is dominated by some primitive recursive function.

5. Show that

$$T = \{x \mid \varphi^{(1)}_x \text{ is total}\}$$

and

$$E = \{x \mid \varphi^{(1)}_x \text{ is the empty function}\}$$

are nonrecursive.

Solution: By the Enumeration and Parametrization theorems, we can find a primitive recursive function f such that

$$\varphi^{(1)}_{f(x)}(y) \;\simeq\; \varphi^{(1)}_x(x)$$

for all x and y. Then $x \in K$ implies $f(x) \in T$, while $x \notin K$ implies $f(x) \in E$. Thus f reduces K to T and to the complement of E.

6. Prove Rice's Theorem: If $\mathcal{C}$ is any nontrivial class of 1-place partial recursive functions, then the index set $I_{\mathcal{C}} = \{x \mid \varphi_x^{(1)} \in \mathcal{C}\}$ is nonrecursive.

   Solution: Let x_0 be an index of the empty function, and let x_1 be an index such that $\varphi_{x_1}^{(1)} \in \mathcal{C}$ if and only if $\varphi_{x_0}^{(1)} \notin \mathcal{C}$. By the Enumeration and Parametrization theorems, we can find a primitive recursive function f such that

$$\varphi_{f(x)}^{(1)}(y) \simeq \begin{cases} \varphi_{x_1}^{(1)}(y) & \text{if } \varphi_x^{(1)}(x) \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

   for all x and y. Thus $x \in K$ implies $\varphi_{f(x)}^{(1)} = \varphi_{x_1}^{(1)}$, while $x \notin K$ implies $\varphi_{f(x)}^{(1)} = \varphi_{x_0}^{(1)}$. Thus f reduces K either to $I_{\mathcal{C}}$ (if $\varphi_{x_1}^{(1)} \in \mathcal{C}$) or to the complement of $I_{\mathcal{C}}$ (if $\varphi_{x_1}^{(1)} \notin \mathcal{C}$). In either case it follows that $I_{\mathcal{C}}$ is not recursive.

7. Show that a predicate $P \subseteq \mathbf{N}^k$ is Δ_2^0 if and only if χ_P is a limit-recursive function.

   Solution:

   For simplicity, let x be an abbreviation for $x_1, \ldots, x_k$.

   First assume that χ_P is limit-recursive, say

$$\chi_P(x) \;=\; \lim_n f(n, x)$$

   for all x, where $f(n, x)$ is a recursive function. Then we have

$$P(x) \;\equiv\; \exists m \forall n (n > m \rightarrow f(n, x) = 1)$$

   and

$$\neg P(x) \;\equiv\; \exists m \forall n (n > m \rightarrow f(n, x) = 0)$$

   so P is Δ_2^0.

   For the converse, assume that P is Δ_2^0, say

$$P(x) \;\equiv\; \exists y \forall z R_1(x, y, z)$$

   and

$$\neg P(x) \;\equiv\; \exists y \forall z R_0(x, y, z)$$

where R_1 and R_0 are primitive recursive predicates. Define $g(n,x) =$ the least $y < n$ such that either $\forall z < n\, R_1(x,y,z)$ or $\forall z < n\, R_0(x,y,z)$ or both, if such a y exists, and $g(n,x) = n$ otherwise. Thus $g(n,x)$ is a primitive recursive function, and it is easy to see that for all x, $g(x) = \lim_n g(n,x)$ exists, and $g(x) =$ the least y such that $\forall z R_1(x,y,z)$ or $\forall z R_0(x,y,z)$. Now define $h(n,x) = 1$ if $\forall z < n\, R_1(x, g(n,x), z)$, and $h(n,x) = 0$ otherwise. Thus $h(n,x)$ is again a primitive recursive function, and for all x, $h(x) = \lim_n h(n,x)$ exists. Moreover $P(x)$ implies $h(x) = 1$, and $\neg P(x)$ implies $h(x) = 0$. Thus χ_P is limit-recursive. This completes the proof.

8. Write down a sentence expressing Goldbach's Conjecture: every even number > 2 is the sum of two primes. Write down a formula defining the function $\lambda xy\,[\,\text{least common multiple of } x \text{ and } y\,]$.

9. Prove König's theorem: If κ_i, λ_i, $i \in I$ are cardinals and $\kappa_i < \lambda_i$ for all $i \in I$, then $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$.

10. Prove Zorn's lemma: If $\mathcal{S}$ is a set of sets which is closed under unions of chains, then $\mathcal{S}$ has a maximal element.

11. Show that $2^{\aleph_0} \neq \aleph_\omega$. (Hint: Use König's theorem.) Generalize.

12. Prove that the real number system (defined in terms of Cauchy sequences) is complete, *i.e.*, every nonempty bounded set of real numbers has a least upper bound.

13. Prove the Generalized Löwenheim-Skolem theorem: Let κ be an infinite cardinal, let (A, E) be a relational structure with $|A| \geq \kappa$, and let $X \subseteq A$ be such that $|X| \leq \kappa$. Then there exists an elementary substructure (A', E') of (A, E) such that $X \subseteq A'$ and $|A'| = \kappa$.

14. Recall that $L_\omega = R_\omega = \{\text{hereditarily finite sets}\}$. For $X \subseteq \omega$, show that $X \in L_{\omega+1}$ if and only if X is arithmetically definable, *i.e.* belongs to the arithmetical hierarchy, *i.e.* $X \in \Sigma^0_k$ for some $k \in \mathbf{N}$.