# Math 558 – Spring 2004 – Homework

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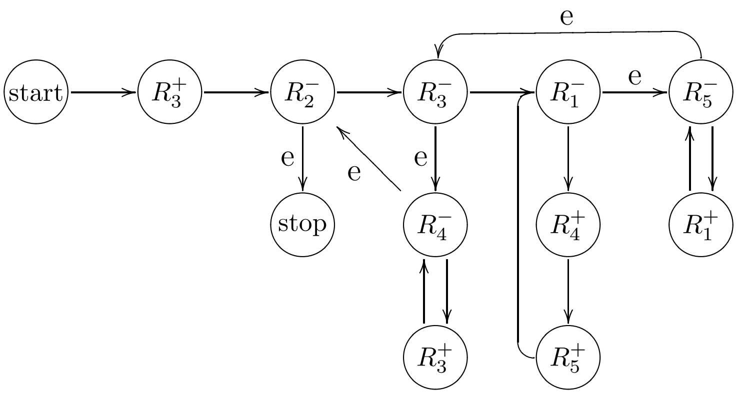
May 3, 2004

1. Show that the function $\lambda n \left( n\text{th digit of } \sqrt{2} \right)$ is primitive recursive.

   *Solution.* The $n$th digit of $\sqrt{2}$ is $f(n) = \mathrm{Rem}(g(n), 10)$, where $g(n) =$ the least $x < 4 \cdot 10^{2n}$ such that $(x+1)^2 > 2 \cdot 10^{2n}$.

2. Write a register machine program which computes the exponential function $\lambda xy\,(x^y)$. Note that $x^0 = 1$ for all $x$.

   *Solution.*



3. For all $n$ let $A_n$ be the $n$th Ackermann branch, defined by

$$
\begin{aligned}
A_0(x) &= 2x, \\
A_{n+1}(x) &= \underbrace{A_n \cdots A_n}_{x}(1).
\end{aligned}
$$

   Thus $A_0(x) = 2x$, $A_1(x) = 2^x$, $A_2(x) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (height $x$), etc.

   Note that, for each $n$, $A_n$ is primitive recursive. In fact, $A_{n+1}$ can be defined by primitive recursion using $A_n$ as

$$
\begin{aligned}
A_{n+1}(0) &= 1, \\
A_{n+1}(x+1) &= A_n(A_{n+1}(x)).
\end{aligned}
$$

(a) Show that $A_n(1) = 2$, $A_n(2) = 4$, and $A_{n+1}(3) = A_n(4)$ for all $n$. Compute $A_n(x)$ for all $n, x$ with $n + x \leq 8$.

*Solution.*

For all $n$ we have $A_{n+1}(1) = A_n(1)$, hence by induction $A_n(1) = A_0(1) = 2$. Also $A_{n+1}(2) = A_n(A_n(1)) = A_n(2)$, hence by induction $A_n(2) = A_0(2) = 4$. Also, for all $n$ and $x$ we have $A_{n+1}(x+1) = A_n(A_{n+1}(x))$, in particular $A_{n+1}(3) = A_n(A_{n+1}(2)) = A_n(4)$. Table 1 shows $A_n(x)$ for small values of $n, x$.

Table 1: The Ackermann branches.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $A_0$ | 0 | 2 | 4 | 6 | 8 | 10 |
| $A_1$ | 1 | 2 | 4 | 8 | 16 | 32 |
| $A_2$ | 1 | 2 | 4 | 16 | $2^{16}$ | $2^{2^{16}}$ |
| $A_3$ | 1 | 2 | 4 | $2^{16}$ | $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (height $2^{16}$) | $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (height $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (height $2^{16}$)) |
| $A_4$ | 1 | 2 | 4 | $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (height $2^{16}$) | $A_3(2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (height $2^{16}$)) | |
| $A_5$ | 1 | 2 | 4 | $A_3(2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (height $2^{16}$)) | | |
| $A_6$ | 1 | 2 | 4 | | | |

(b) Prove the following:

i. $A_n(x+1) > A_n(x) > x$ for all $x \geq 1$ and all $n$.

ii. $A_{n+1}(x) \geq A_n(x+1)$ for all $x \geq 3$ and all $n$.

iii. For each primitive recursive function $f(x_1, \ldots, x_k)$ there exists $n$ such that $A_n$ *covers* $f$, i.e.,

$$f(x_1, \ldots, x_k) \leq A_n(\max(3, x_1, \ldots, x_k))$$

for all $x_1, \ldots, x_k$.

iv. The 1-place function $\lambda x\,(A_x(x))$ is not primitive recursive.

v. The 2-place function $\lambda xy\,(A_x(y))$ is not primitive recursive.

*Solution.* First we prove $A_n(x+1) > A_n(x) > x$ for $x \geq 1$, by induction on $n$. For $n = 0$ we have $A_0(x+1) = 2x+2 > 2x = A_0(x)$ for all $x$, and $A_0(x) = 2x > x$ for $x \geq 1$. For $n+1$ and $x \geq 1$ we have $A_{n+1}(x+1) = A_n(A_{n+1}(x)) > A_{n+1}(x)$ by inductive hypothesis. Thus $A_{n+1}$ is strictly monotone. Since $A_{n+1}(0) > 0$, it follows that $A_{n+1}(x) > x$ for all $x$.

Next we prove $A_{n+1}(x) \geq A_n(x+1)$ for $x \geq 3$, by induction on $x$. For $x = 3$ we have $A_{n+1}(3) = A_n(4)$ as noted above, and inductively $A_{n+1}(x+1) = A_n(A_{n+1}(x)) \geq A_n(A_n(x+1)) \geq A_n(x+2)$, since $A_n$ is strictly monotone and $A_n(x+1) \geq x+2$ by what has already been proved.

Next we prove that each primitive recursive function is covered by $A_n$ for some $n$. We prove this by induction on the class of primitive recursive functions. We begin by noting that the initial functions are covered by $A_0$.

Suppose $f$ is obtained by generalized composition, say

$$f(x_1, \ldots, x_k) = h(g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k)).$$

Let $n$ be such that $A_n$ covers $h$ and $A_{n+1}$ covers $g_1, \ldots, g_m$. We then have

$$\begin{aligned} f(x_1, \ldots, x_k) &= h(g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k)) \\ &\leq A_n(\max(3, g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k))) \\ &\leq A_n(A_{n+1}(\max(3, x_1, \ldots, x_k))) \\ &= A_{n+1}(\max(3, x_1, \ldots, x_k) + 1) \\ &\leq A_{n+2}(\max(3, x_1, \ldots, x_k)), \end{aligned}$$

i.e., $A_{n+2}$ covers $f$.

Suppose $f$ is obtained by primitive recursion, say

$$\begin{aligned} f(0, x_1, \ldots, x_k) &= g(x_1, \ldots, x_k), \\ f(y+1, x_1, \ldots, x_k) &= h(y, f(y, x_1, \ldots, x_k), x_1, \ldots, x_k). \end{aligned}$$

Let $n$ be such that $A_n$ covers $h$ and $A_{n+1}$ covers $g$. We first claim that

$$f(y, x_1, \ldots, x_k) \leq A_{n+1}(y + \max(3, x_1, \ldots, x_k))$$

for all $y, x_1, \ldots, x_k$. We prove this by induction on $y$. For $y = 0$ we have $f(0, x_1, \ldots, x_k) = g(x_1, \ldots, x_k) \leq A_{n+1}(\max(3, x_1, \ldots, x_k))$. For the inductive step we have

$$\begin{aligned} f(y+1, x_1, \ldots, x_k) &= h(y, f(y, x_1, \ldots, x_k), x_1, \ldots, x_k) \\ &\leq A_n(\max(3, y, f(y, x_1, \ldots, x_k), x_1, \ldots, x_k) \\ &\leq A_n(\max(3, y, A_{n+1}(y + \max(3, x_1, \ldots, x_k)), x_1, \ldots, x_k) \\ &= A_n(A_{n+1}(y + \max(3, x_1, \ldots, x_k))) \\ &= A_{n+1}(y + 1 + \max(3, x_1, \ldots, x_k)) \end{aligned}$$

and this proves our claim. We then have

$$\begin{aligned} f(y, x_1, \ldots, x_k) &\leq A_{n+1}(y + \max(3, x_1, \ldots, x_k)) \\ &\leq A_{n+1}(2 \max(3, y, x_1, \ldots, x_k)) \\ &\leq A_{n+1}(A_{n+2}(\max(3, y, x_1, \ldots, x_k))) \\ &= A_{n+2}(\max(3, y, x_1, \ldots, x_k) + 1) \\ &\leq A_{n+3}(\max(3, y, x_1, \ldots, x_k)), \end{aligned}$$

i.e., $A_{n+3}$ covers $f$. This completes the proof that each primitive recursive function is covered by $A_n$ for some $n$.

Now, if $A_x(x)$ were primitive recursive, then $A_x(x)+1$ would be primitive recursive, hence covered by $A_n$ for some $n \geq 3$. But then in particular $A_n(n)+1 \leq A_n(\max(3,n)) = A_n(n)$, a contradiction. Thus the 1-place function $A_x(x)$ is not primitive recursive. It follows immediately that the 2-place function $A_x(y)$ is not primitive recursive.

(c) (Extra Credit) Show that the 3-place relation

$$\{\langle x,y,z\rangle \mid A_x(y)=z\}$$

is primitive recursive. Use this to prove that $\lambda xy\,(A_x(y))$ is recursive. Hence $\lambda x\,(A_x(x))$ is recursive.

*Solution.* For all $x,y>0$ we have

$$0<y<A_x(y)=A_{x-1}(A_x(y-1))=A_{x-1}(y')$$

where $y'=A_x(y-1)$. Since $A_{x-1}(y')=A_x(y)\geq 2$, it follows that $0<y'<A_{x-1}(y')=A_x(y)$. Repeating this step $x$ times, we obtain a finite sequence $y_0,y_1,y_2,\ldots,y_x$ starting with $y$ such that

$$A_x(y)=A_x(y_0)=A_{x-1}(y_1)=A_{x-2}(y_2)=\cdots=A_0(y_x)=2y_x,$$

and each of $y_0,y_1,\ldots,y_x$ is $>0$ and $<A_x(y)$. Moreover, if $y>2$ then we also have $x<A_x(y)$. Thus the 3-place predicate $A_x(y)=z$ can be defined by course-of-values recursion on $z$ as follows:

$A_x(y)=z$ if and only if

$(x=0\wedge z=2y)\vee$

$(x>0\wedge y=0\wedge z=1)\vee$

$(x>0\wedge y=1\wedge z=2)\vee$

$(x>0\wedge y=2\wedge z=4)\vee$

$(x>0\wedge y>2\wedge x<z\wedge \exists y_0,y_1,\ldots,y_x<z$

$(y_0=y\wedge \forall i<x\,(y_{i+1}=A_{x-i}(y_i-1))\wedge z=2y_x))$.

Actually, the function being defined by primitive recursion is

$$a(w)=\prod\{p_{2^x3^y5^z}\mid A_x(y)=z\wedge x,y,z<w\}.$$

In any case, it follows that the 3-place predicate $A_x(y)=z$ is primitive recursive.

Applying the least number operator, we see that the 2-place function $A_x(y)$ is recursive. It follows immediately that the 1-place function $A_x(x)$ is recursive.

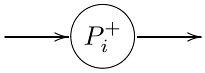
4. Given a $k$-place partial recursive function $\psi(x_1,\ldots,x_k)$, show that there is a 1-place partial recursive function $\psi^*(z)$ such that

$$\psi^*(p_1^{x_1}\cdots p_k^{x_k}) \;\simeq\; p_{k+1}^{\psi(x_1,\ldots,x_k)}$$

for all $x_1,\ldots,x_k$, and $\psi^*(z)$ is computable by a register machine using only two registers, $R_1$ and $R_2$.

*Solution.* We begin with a register machine program $\mathcal{P}$ which computes $\psi(x_1,\ldots,x_k)$. Let $P_1,\ldots,P_k,P_{k+1},\ldots,P_t$ be the registers used in $\mathcal{P}$. We may safely assume that, whenever $\mathcal{P}(x_1,\ldots,x_k)$ halts, it leaves all registers except $P_{k+1}$ empty.

We transform $\mathcal{P}$ into a program $\mathcal{R}$ which uses only two registers, $R_1$ and $R_2$. The idea is that, if $P_1,\ldots,P_t$ contain $z_1,\ldots,z_t$ respectively, then $R_1$ contains $z = p_1^{z_1}\cdots p_t^{z_t}$, while $R_2$ contains 0. Incrementing (decrementing) $P_i$ corresponds to multiplication (division) by $p_i$. Each instruction in $\mathcal{P}$ is replaced by a corresponding set of instructions in $\mathcal{R}$.

We replace  in $\mathcal{P}$ by Figure 1 in $\mathcal{R}$.

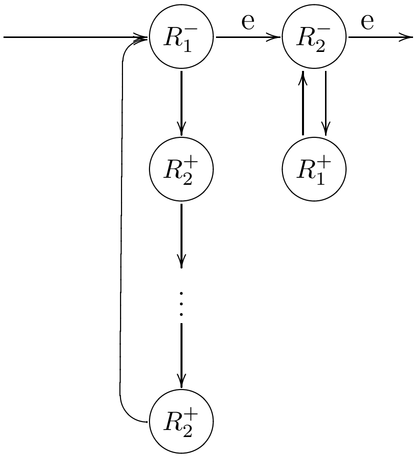
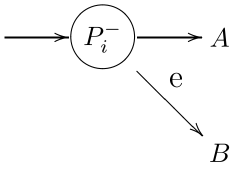
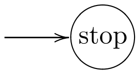


Figure 1: Incrementing $P_i$. The number of $R_2^+$ instructions is $p_i$.

We replace  in $\mathcal{P}$ by Figure 2 in $\mathcal{R}$.

We replace  in $\mathcal{P}$ by Figure 3 in $\mathcal{R}$.

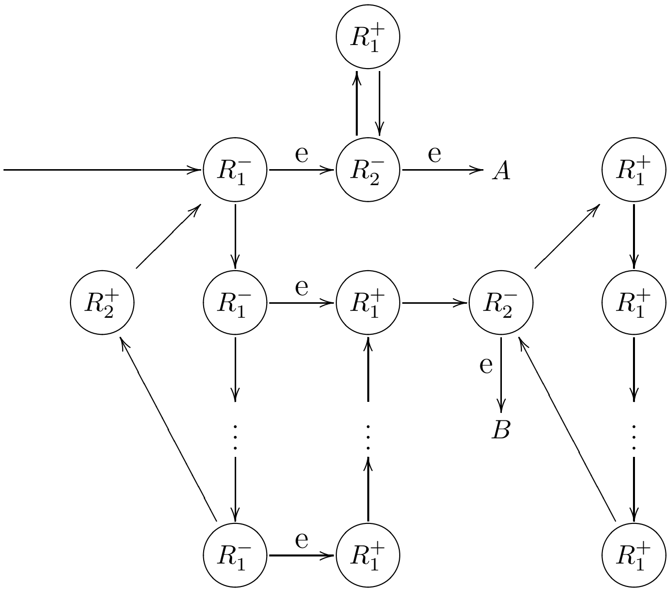


Figure 2: Decrementing $P_i$. The number of $R_1^-$ instructions is $p_i$.

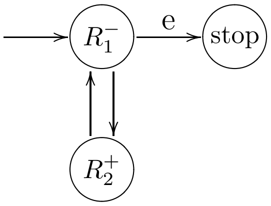


Figure 3: Stopping.

5. Let $T = \{\text{indices of total recursive functions}\} = \{e \mid \forall x\, \varphi_e^{(1)}(x)\downarrow\}$. Let $E = \{\text{indices of the empty function}\} = \{e \mid \forall x\, \varphi_e^{(1)}(x)\uparrow\}$. Show that $T$ and $E$ are not recursive.

   *Solution.* By the Enumeration and Parametrization theorems, we can find a primitive recursive function $f$ such that $\varphi_{f(x)}^{(1)}(y) \simeq \varphi_x^{(1)}(x)$ for all $x$ and $y$. Then $x \in K$ implies $f(x) \in T$, while $x \notin K$ implies $f(x) \in E$. Thus, since $T$ and $E$ are disjoint, $f$ reduces $K$ to $T$ and to the complement of $E$. Since $K$ is non-recursive, it follows that $T$ and $E$ are non-recursive.

6. Let $\mathcal{P}$ be the class of 1-place partial recursive functions. For $\mathcal{C} \subseteq \mathcal{P}$, define $I_{\mathcal{C}}$ to be the set of indices of functions in $\mathcal{C}$, i.e.,

$$I_{\mathcal{C}} = \{e \in \mathbb{N} \mid \varphi_e^{(1)} \in \mathcal{C}\}.$$

   Show that if $\emptyset \neq \mathcal{C} \neq \mathcal{P}$ then $I_{\mathcal{C}}$ is nonrecursive. (This result is known as Rice's Theorem.)

   *Solution.* Let $e_0$ be an index of the empty function. Let $e_1$ be an index such that $\varphi_{e_1}^{(1)} \in \mathcal{C}$ if and only if $\varphi_{e_0}^{(1)} \notin \mathcal{C}$. By the Enumeration and Parametrization theorems, we can find a primitive recursive function $f$ such that

$$\varphi_{f(x)}^{(1)}(y) \simeq \begin{cases} \varphi_{e_1}^{(1)}(y) & \text{if } \varphi_x^{(1)}(x)\downarrow, \\ \uparrow & \text{otherwise,} \end{cases}$$

   for all $x$ and $y$. Thus $x \in K$ implies $\varphi_{f(x)}^{(1)} = \varphi_{e_1}^{(1)}$, while $x \notin K$ implies $\varphi_{f(x)}^{(1)} = \varphi_{e_0}^{(1)}$. Thus $f$ reduces $K$ either to $I_{\mathcal{C}}$ (if $\varphi_{e_1}^{(1)} \in \mathcal{C}$) or to the complement of $I_{\mathcal{C}}$ (if $\varphi_{e_1}^{(1)} \notin \mathcal{C}$). In either case it follows that $I_{\mathcal{C}}$ is not recursive.

7. Find $m$ and $n$ such that $m \neq n$ and $\varphi_m^{(1)}(0) = n$ and $\varphi_n^{(1)}(0) = m$.

   *Solution.*

   By the Parametrization Theorem, let $f$ be a 1-place primitive recursive function such that $\varphi_{f(x)}^{(1)}(y) = x$ for all $x, y$. The construction of $f$ in the proof of the Parametrization Theorem shows that $f(x) > x$ for all $x$. By the Recursion Theorem, let $e$ be such that $\varphi_e^{(1)}(y) \simeq f(e)$ for all $y$. In particular we have $\varphi_e^{(1)}(0) \simeq f(e)$, $\varphi_{f(e)}^{(1)}(0) = e$, and $f(e) > e$. Thus we may take $m = e$ and $n = f(e)$.

8. A function $f : \mathbb{N}^k \to \mathbb{N}$ is said to be *limit recursive* if there exists a recursive function $g : \mathbb{N}^{k+1} \to \mathbb{N}$ such that, for all $x_1, \ldots, x_k$, $f(x_1, \ldots, x_k) = \lim_s g(x_1, \ldots, x_k, s)$.

Let $P$ be a $k$-place predicate. Show that $P$ is $\Delta^0_2$ if and only if $\chi_P$ is limit recursive.

*Solution.*

For simplicity, let $x$ be an abbreviation for $x_1, \ldots, x_k$.

First assume that $\chi_P$ is limit recursive, say

$$\chi_P(x) \;=\; \lim_n f(n,x)$$

for all $x$, where $f(n,x)$ is a recursive function. Then we have

$$P(x) \;\equiv\; \exists m \, \forall n \, (n \geq m \Rightarrow f(n,x) = 1)$$

and

$$\neg P(x) \;\equiv\; \exists m \, \forall n \, (n \geq m \Rightarrow f(n,x) = 0)$$

so $P$ is $\Delta^0_2$.

For the converse, assume that $P$ is $\Delta^0_2$, say

$$P(x) \;\equiv\; \exists y \, \forall z \, R_1(x,y,z)$$

and

$$\neg P(x) \;\equiv\; \exists y \, \forall z \, R_0(x,y,z)$$

where $R_1$ and $R_0$ are primitive recursive predicates. Using the bounded least number operator, define $g(n,x) =$ the least $y < n$ such that either $\forall z < n \, R_1(x,y,z)$ or $\forall z < n \, R_0(x,y,z)$ or both, if such a $y$ exists, and $g(n,x) = n$ otherwise. Thus $g(n,x)$ is a primitive recursive function, and it is easy to see that, for all $x$, $g(x) = \lim_n g(n,x)$ exists and is equal to the least $y$ such that $\forall z \, R_1(x,y,z)$ or $\forall z \, R_0(x,y,z)$. Now define $h(n,x) = 1$ if $\forall z < n \, R_1(x, g(n,x), z)$, and $h(n,x) = 0$ otherwise. Thus $h(n,x)$ is again a primitive recursive function, and for all $x$, $h(x) = \lim_n h(n,x)$ exists. Moreover $P(x)$ implies $h(x) = 1$, and $\neg P(x)$ implies $h(x) = 0$. Thus $\chi_P$ is limit recursive. This completes the proof.

9. (a) Show that $\operatorname{card}(\mathbb{R}) = 2^{\aleph_0}$, where $\mathbb{R}$ is the set of real numbers.
   (b) Show that $\operatorname{card}(\mathbb{R}^{\mathbb{R}}) = 2^{2^{\aleph_0}}$. (Recall that $\mathbb{R}^{\mathbb{R}}$ is the set of functions from $\mathbb{R}$ into $\mathbb{R}$.)
   (c) Show that $\operatorname{card}(C(\mathbb{R},\mathbb{R})) = 2^{\aleph_0}$, where $C(\mathbb{R},\mathbb{R})$ is the set of continuous functions from $\mathbb{R}$ into $\mathbb{R}$.

10. Let $\langle \kappa_i \mid i \in I \rangle$ and $\langle \lambda_i \mid i \in I \rangle$ be indexed families of cardinals with the same index set, $I$. Show that, if $\kappa_i < \lambda_i$ for all $i \in I$, then

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i.$$

This result is known as König's Theorem.

11. Show that the operations of ordinal arithmetic may be defined by transfinite recursion as

$$\begin{aligned} \alpha+\beta &= \{\alpha\}\cup\bigcup\{(\alpha+\gamma)+1 \mid \gamma<\beta\}, \\ \alpha\cdot\beta &= \bigcup\{(\alpha\cdot\gamma)+\alpha \mid \gamma<\beta\}, \\ \alpha^{\beta} &= \{1\}\cup\bigcup\{(\alpha^{\gamma})\cdot\alpha \mid \gamma<\beta\}. \end{aligned}$$

Alternatively, letting $\delta$ denote a limit ordinal, we may define the operations of ordinal arithmetic by

$$\begin{aligned} \alpha+0 &= \alpha \\ \alpha+(\beta+1) &= (\alpha+\beta)+1 \\ \alpha+\delta &= \bigcup\{\alpha+\gamma \mid \gamma<\delta\} \end{aligned}$$

$$\begin{aligned} \alpha\cdot 0 &= 0 \\ \alpha\cdot(\beta+1) &= (\alpha\cdot\beta)+\alpha \\ \alpha\cdot\delta &= \bigcup\{\alpha\cdot\gamma \mid \gamma<\delta\} \end{aligned}$$

$$\begin{aligned} \alpha^{0} &= 1 \\ \alpha^{\beta+1} &= (\alpha^{\beta})\cdot\alpha \\ \alpha^{\delta} &= \bigcup\{\alpha^{\gamma} \mid \gamma<\delta\}. \end{aligned}$$

12. Let $\kappa$ be an uncountable cardinal. The *cofinality* of $\kappa$ is defined to be the least cardinal $\lambda$ such that $\kappa$ can be written as the sum of $\lambda$ cardinals each $< \kappa$. We write $\mathrm{cf}(\kappa) =$ the cofinality of $\kappa$.

    (a) Show that $\mathrm{cf}(\kappa)$ is an infinite regular cardinal, and $\mathrm{cf}(\kappa) \le \kappa$.

    (b) Show that $\kappa$ is singular if and only if $\mathrm{cf}(\kappa) < \kappa$.

    (c) Show that $\kappa^{\mathrm{cf}(\kappa)} > \kappa$.

    (d) Show that, for all infinite cardinals $\lambda$, $\mathrm{cf}(2^{\lambda}) > \lambda$.

    (e) In particular, show that $\mathrm{cf}(2^{\aleph_0}) > \aleph_0$.

13. Let $\kappa$ be an inaccessible cardinal. Show that there exists a cardinal $\lambda < \kappa$ such that $R_\lambda$ is an elementary submodel of $R_\kappa$. Show that we may obtain $\lambda$ with the additional property $\mathrm{cf}(\lambda) = \aleph_0$.