Math 558 – Homework #5

April 13, 2007

1. Given an ordinal $\delta > 0$, show that the following assertions are pairwise equivalent.

   (a) $\alpha + \delta = \delta$ for all $\alpha < \delta$.

   (b) $\alpha + \beta < \delta$ for all α, $\beta < \delta$.

   (c) $\delta = \omega^\gamma$ for some $\gamma \leq \delta$.

   An ordinal with these properties is said to be *additively indecomposable.*

2. Given an ordinal γ, prove that there is one and only one way to write γ in the form
$$\gamma = \delta_1 + \cdots + \delta_n$$
where $n \in \mathbb{N}$ and $\delta_1 \geq \cdots \geq \delta_n > 0$ are additively indecomposable.

3. Prove that $\operatorname{card}(\mathbb{R}^\mathbb{N}) = 2^{\aleph_0}$ and $\operatorname{card}(\mathbb{R}^\mathbb{R}) = 2^{2^{\aleph_0}}$.

4. Prove that there exist arbitrarily large uncountable cardinals λ such that $\lambda = \aleph_\lambda$.

5. Given an uncountable cardinal λ, define the *cofinality* of λ, written $\operatorname{cf}(\lambda)$, to be the smallest cardinal κ such that $\lambda = \sum_{i \in I} \lambda_i$ for some indexed set of cardinals $\lambda_i < \lambda$, $i \in I$, where $|I| = \kappa$.

   Prove the following.

   (a) $\operatorname{cf}(\lambda) \leq \lambda$. $\operatorname{cf}(\lambda)$ is regular.

   (b) λ is regular if and only if $\operatorname{cf}(\lambda) = \lambda$.

   (c) λ is singular if and only if $\operatorname{cf}(\lambda) < \lambda$.

   (d) $\lambda^{\operatorname{cf}(\lambda)} > \lambda$. (Hint: Use König's Theorem.)

   (e) For any infinite cardinal κ we have $\operatorname{cf}(\mu^\kappa) > \kappa$ for all $\mu > 1$. In particular, $\operatorname{cf}(2^\kappa) > \kappa$.

6. Let κ be an infinite regular cardinal. Prove that there exist arbitrarily large strong limit cardinals λ such that $\operatorname{cf}(\lambda) = \kappa$. Moreover, for all such λ we have $\lambda^\mu = \lambda$ for all μ in the interval $1 \leq \mu < \kappa$.